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COMPARISON SURFACES FOR THE WILLMORE PROBLEM

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The infimum of the conformally invariant functional $W = \int H^2$ is estimated for each regular homotopy class of immersed surfaces in $\mathbf{R}^3$. Consequently, we obtain rather sharp bounds on the maximum multiplicity and branching order of a W-minimizing surface. In the case of $\mathbf{R}P^2$ we provide an example of a symmetric W-minimizing Boy's surface ($W = 12\pi$)—as well as symmetric static surfaces of higher index—thereby solving part of the Willmore problem.

0. Introduction. This paper addresses the well-known variational problem posed by T. J. Willmore [**WT**] in 1965:

Find the minimum for the squared-mean-curvature integral

$$W(M) = \int_M H^2\, da$$

among compact embedded surfaces $M \subset \mathbf{R}^3$ of a given genus.

Willmore noted that $W(M) \geq 4\pi$, with equality only for *round spheres*. He also found a torus $M_1 \subset \mathbf{R}^3$ with $W(M_1) = 2\pi^2$ and conjectured this value to be the minimum among embedded tori. Although Willmore's conjecture remains unresolved, at least his example serves as a *comparison surface*, showing that the infimum of W among embedded tori is not greater than $2\pi^2 < 8\pi$.

This is our starting point: for each genus g we exhibit a comparison surface $M_g \subset \mathbf{R}^3$ with $W(M_g) < 8\pi$. More generally, we consider the Willmore problem for *immersed* surfaces $M \looparrowright \mathbf{R}^3$. A path of immersed surfaces is a *regular homotopy*, and the path component—or *regular homotopy class*—of $M \looparrowright \mathbf{R}^3$ is denoted by $[M]$. We will construct (in §5) appropriate comparison surfaces to deduce the following

MAIN THEOREM. *The infimum $W_{[M]}$ for W over any regular homotopy class $[M]$ of compact immersed surfaces $M \looparrowright \mathbf{R}^3$ satisfies*

$$W_{[M]} < 20\pi,$$

with the best upper estimates known given in §5. In particular, the infimum of W among compact immersed surfaces of a given topological