DISTANCE BETWEEN UNITARY ORBITS IN VON NEUMANN ALGEBRAS

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Dedicated to Professor Shozo Koshi on his 60th birthday

Let $\mathcal{M}$ be a semifinite factor. For normal operators $x$ and $y$ in $\mathcal{M}$, introducing the spectral distance $\delta(x, y)$, we show that $\delta(x, y) \geq \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq c^{-1}\delta(x, y)$ with a universal constant $c$, where $\text{dist}(\mathcal{U}(x), \mathcal{U}(y))$ denotes the distance between the unitary orbits $\mathcal{U}(x)$ and $\mathcal{U}(y)$. The equality $\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = \delta(x, y)$ holds in several cases. Submajorizations are established concerning the spectral scales of $\tau$-measurable selfadjoint operators affiliated with $\mathcal{M}$. Using these submajorizations, we obtain the formulas of $L^p$-distance and anti-$L^p$-distance between unitary orbits of $\tau$-measurable selfadjoint operators in terms of their spectral scales. Furthermore, the formulas of those distances in Haagerup $L^p$-spaces are obtained when $\mathcal{M}$ is a type III$_1$ factor. The appendix by H. Kosaki is the generalized Powers-Størmer inequality in Haagerup $L^p$-spaces.

Introduction. It is an interesting problem in matrix theory to estimate distances between unitary orbits of matrices by their eigenvalues. Let $A$ and $B$ be $n \times n$ normal matrices whose eigenvalues are $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively, with multiplicities counted. Let $\text{dist}(\mathcal{U}(A), \mathcal{U}(B))$ denote the distance between the unitary orbits $\mathcal{U}(A)$ and $\mathcal{U}(B)$. The optimal matching distance between the eigenvalues of $A$ and $B$ is given by

$$\delta(A, B) = \min_{\pi} \max_i |\alpha_i - \beta_{\pi(i)}|,$$

where $\pi$ runs over all permutations of $\{1, \ldots, n\}$. Then

$$\text{dist}(\mathcal{U}(A), \mathcal{U}(B)) \leq \delta(A, B)$$

is immediate. Bhatia, Davis and McIntosh [9] proved that

$$\text{dist}(\mathcal{U}(A), \mathcal{U}(B)) \geq c^{-1}\delta(A, B)$$

with a universal constant $c$. A difficult and still open conjecture is that $\text{dist}(\mathcal{U}(A), \mathcal{U}(B)) = \delta(A, B)$ holds for every pair of normal matrices $A$ and $B$ (i.e. $c = 1$). But this equality was proved to hold for several classes of normal matrices (see [7, 10, 21, 41, 45]). The analogous