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HARMONIC MEASURES SUPPORTED ON CURVES

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Let Ω_1 and Ω_2 be two disjoint, simply connected domains in the plane, and let ω_1 and ω_2 be harmonic measures associated to Ω_1 and Ω_2. We present necessary and sufficient conditions for ω_1 and ω_2 to be mutually singular.

1. Introduction. Let Γ be a Jordan curve in C and let Ω_1 and Ω_2 be the two simply connected domains complementary to Γ. For each domain Ω_j fix a point $z_j \in \Omega_j$ and let ω_j be the harmonic measure for z_j relative to Ω_j. In this paper we discuss when the two measures are singular, $\omega_1 \perp \omega_2$, i.e. when there are disjoint sets E_1, E_2 such that $\omega_j(E_j) = 1$, $j = 1, 2$. If Γ is a Jordan arc, Γ^c consists of only one domain Ω, but since Γ has two sides there are two measures ω_1, ω_2 which give the harmonic measure of sets on each of the two sides of Γ. Again it makes sense to ask whether $\omega_1 \perp \omega_2$.

If Γ is a Jordan curve or arc and $z_0 \in \Gamma$ we say that Γ has a tangent at z_0 if there is θ_0 with the property that for all $\varepsilon > 0$ there is $r > 0$ such that whenever $z \in \Gamma$ and $|z - z_0| < r$, either $|\theta_0 - \arg(z - z_0)| < \varepsilon$ or $|\theta_0 + \pi - \arg(z - z_0)| < \varepsilon$. We denote by T the collection of all tangent points on Γ. When Γ is a Jordan curve we also say that $z_0 \in T_1$ if there is a unique θ_0 (mod 2π) with the property that for all $\varepsilon > 0$ there is $r > 0$ such that

$$\{z : 0 < |z - z_0| < r,\ |\theta_0 - \arg(z - z_0)| < \pi/2 - \varepsilon\} \subset \Omega_1.$$

T_1 is called the set of inner tangent points with respect to Ω_1. With T_2 similarly defined one sees that $T = T_1 \cap T_2$. If Γ is a Jordan arc T_1 and T_2 are similarly defined. Finally, we denote one dimensional Hausdorff measure by Λ_1.

THEOREM. *Suppose Γ is a Jordan curve or arc. Then $\omega_1 \perp \omega_2$ if and only if $\Lambda_1(T) = 0$.*

Let $A(\Gamma)$ denote the class of all bounded continuous functions on the Riemann sphere which are holomorphic off Γ. In **[4]** Browder and Wermer proved that $A(\Gamma)$ is a Dirichlet algebra if and only if $\omega_1 \perp \omega_2$.