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COMMUTATIVE SUBALGEBRAS OF THE RING OF DIFFERENTIAL OPERATORS ON A CURVE

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Let X denote an irreducible affine algebraic curve over the complex numbers. Let $\mathscr{O}(X)$ be the ring of regular functions on X. Denote by $\mathscr{D}(X)$ the ring of differential operators on X. We wish to characterize $\mathscr{O}(X)$ as a ring theoretic invariant of $\mathscr{D}(X)$. It is proved that $\mathscr{O}(X)$ equals the set of all locally ad-nilpotent elements of $\mathscr{D}(X)$ if and only if X is not simply connected. However, for most simply connected curves, we show there exists a maximal commutative subalgebra of $\mathscr{D}(X)$, consisting of locally ad-nilpotent elements, which is not isomorphic to $\mathscr{O}(X)$.

0. Introduction. Let X be a curve, that is, an irreducible affine algebraic curve over $\mathbb{C}$. Write $\mathscr{O}(X)$ for the ring of regular functions on X and $\mathscr{D}(X)$ for the ring of differential operators on X. See [**8**] for the basic definitions and facts about rings of differential operators on curves. This paper is motivated by the following question. If X and Y are curves with $\mathscr{D}(X) \cong \mathscr{D}(Y)$, is $X \cong Y$? Write $\tilde{X}$ for the normalization of X. Stafford [**9**] considers this question for X with $\tilde{X} = \mathbb{A}^1$, the affine line. He shows that $\mathscr{D}(X) \cong \mathscr{D}(\tilde{X})$ if and only if $X = \tilde{X}$. He also shows that if X is the cubic cusp $y^2 = x^3$ and $\tilde{Y} = \mathbb{A}^1$, then $X \cong Y$ if and only if $\mathscr{D}(X) \cong \mathscr{D}(Y)$. Higher dimensional non-isomorphic varieties can have isomorphic rings of differential operators, see Levasseur, Smith and Stafford [**2**].

If $u \in \mathscr{D}(X)$, define $\mathrm{ad}(u) \in \mathrm{End}_{\mathbb{C}}(\mathscr{D}(X))$ by $\mathrm{ad}(u)(v) = [u,v] = uv - vu$. We say u is *locally ad-nilpotent* if for every $v \in \mathscr{D}(X)$ there exists $n \in \mathbb{N}$ with $\mathrm{ad}(u)^n(v) = 0$. Write

$$\mathscr{N}(X) = \{u \in \mathscr{D}(X) | u \text{ is locally ad-nilpotent}\}.$$

Note that if $\vartheta\colon \mathscr{D}(Y) \to \mathscr{D}(X)$ is an isomorphism then $\vartheta(\mathscr{N}(Y)) = \mathscr{N}(X)$. It follows from the definition of $\mathscr{D}(X)$ that $\mathscr{O}(X)$ is a maximal commutative subalgebra of $\mathscr{D}(X)$ and that $\mathscr{O}(X)$ is contained in $\mathscr{N}(X)$. If genus $(X) > 0$ then Makar-Limanov [**3**] shows that $\mathscr{O}(X) = \mathscr{N}(X)$. Hence if $\mathscr{D}(X) \cong \mathscr{D}(Y)$ with genus$(X) > 0$ then $X \cong Y$.

This paper expands on Makar-Limanov's result to prove the following theorem. Let $\pi\colon \tilde{X} \to X$ denote the canonical surjection.