PACIFIC JOURNAL OF MATHEMATICS
Vol. 139, No. 2, 1989

OPERATOR ESTIMATES USING THE SHARP FUNCTION

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Let $f^{\#}$ be the sharp function introduced by Fefferman and Stein. Suppose that T and U are operators acting on the space of Schwartz functions which satisfy the pointwise estimate $(Tf)^{\#}(x) \leq A|Uf(x)|$. Then, on the L^p spaces, the operator norm of T divided by the operator norm of U is bounded by a constant times p. This result allows us to obtain the best possible rate of growth estimate, as $p \to \infty$, on the norms of singular integrals, multipliers, and pseudo-differential operators. These estimates remain valid on weighted L^p spaces defined by an A_∞ weight.

Introduction. Let $\mathscr{S}$ be the space of Schwartz testing functions and suppose that T and U are two operators acting on $\mathscr{S}$. Assume that for all $f \in \mathscr{S}$ we have the pointwise inequality

$$(1) \qquad (Tf)^{\#}(x) \leq A|Uf(x)|,$$

where $g^{\#}$ is the Fefferman and Stein sharp function. It is well known that (1) implies the L^p inequality $\|Tf\|_p \leq C(p)\|Uf\|_p$, $1 < p < \infty$, where the constant $C(p)$ grows exponentially in p. Moreover, these norms can be replaced by a weighted norm as long as the weight satisfies the A_∞ condition.

In fact, inequality (1) implies a much stronger result. We show that as long as Tf satisfies mild growth conditions for all $f \in \mathscr{S}$, the constant $C(p)$ grows linearly in p. This result is then used to prove best possible norm estimates for a large class of singular integral and multiplier operators.

The proof of the main theorem is based on a better understanding of the distribution function of the sharp function. The standard norm estimate for the sharp function is a consequence of what is referred to as a "good-λ inequality." In Lemma 1 we improve the good-λ inequality by showing

$$(2) \quad w(\{x \in \mathbb{R}^n : |f(x)| > Bf^{\#}(x) + \lambda\}) \leq \varepsilon w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}).$$