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ALGEBRAIC INDEPENDENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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We deal with second order algebraic differential equations obtained by equating exact and logarithmic derivatives. Under the assumption that such an equation has no "first integral" (which is proven in particular cases), it is shown that two generic solutions can be algebraically independent only if they satisfy a "very special" relation. Whence is deduced the existence of an infinite algebraically free set of generic solutions over a constant differential field.

1. Introduction. Let P be a differential polynomial with coefficients in an *ordinary differential field* k *of characteristic zero* (in short: d.f.). Suppose that P is of the order $N > 0$ and irreducible (i.e. P is in $k[X, X', \dots, X^{(N)}]$ and is irreducible in this UFD). We are concerned with the following algebraic questions, which are to be made more precise later.

I. *Does the equation* $P = 0$ *admit a "first integral"* (within the frame of differential algebra)?

In this paper, we prove that the answer is negative for equations of order 2 in a certain class; furthermore, if such an equation is "the minimal equation" over k of a non-constant element x (see infra for definitions), then it remains so for x over any d.f. extension K of k over which x is not algebraic (see Theorem 1 and Corollary).

As a consequence, the adjunction of such an element to k introduces no transcendental constant; in classical terms, this means that the general solution of this second order equation is not even parametrized by one arbitrary constant.

II. *Suppose* $P = 0$ *has these properties. What can be said in terms of algebraic independence over* k, *of the solutions of* $P = 0$ *in some differentially closed extension of* k, *e.g. in a differential closure* $\hat{k}$ *of* k? (See Section 3.)

In case k is a finite d.f. extension of the prime d.f. $\mathbb{Q}$, this problem is related to the classification of countable differentially closed fields, which is discussed in **[8]**.