PACIFIC JOURNAL OF MATHEMATICS
Vol. 141, No. 2, 1990

COMPLEMENTATION OF CERTAIN SUBSPACES OF $L_\infty(G)$ OF A LOCALLY COMPACT GROUP

ANTHONY TO-MING LAU AND VIKTOR LOSERT

Let G be a locally compact group, $\mathrm{WAP}(G)$ be the space of continuous weakly almost periodic functions on G and $C_0(G)$ the space of continuous functions on G vanishing at infinity. We prove in this paper, among other things, that if G is infinite and X is any subspace of $\mathrm{WAP}(G)$ (or $\mathrm{CB}(G)$, the space of bounded continuous functions in case G is nondiscrete) containing $C_0(G)$, then X is uncomplemented in $L_\infty(G)$. If G is non-compact, then $\mathrm{WAP}(G)$ is uncomplemented in $\mathrm{LUC}(G)$. Furthermore, $\mathrm{AP}(G)$, the space of continuous almost periodic functions on G, is complemented in $\mathrm{LUC}(G)$ if and only if G/N is compact, where N is the intersection of the kernels of all finite-dimensional continuous unitary representations of G. We also prove that if A is any left translation invariant C^*-subalgebra of $C_0(G)$, then A is the range of a continuous projection commuting with left translations.

1. Introduction and some preliminaries. Let G be a locally compact group and $\mathrm{CB}(G)$ be the space of bounded continuous complex-valued functions on G with supremum norm. Let $\mathrm{LUC}(G)$ denote the space of bounded left uniformly continuous complex-valued functions on G, i.e. all $f \in \mathrm{CB}(G)$ such that the map $g \to l_g f$ from G into $\mathrm{CB}(G)$ is continuous when $\mathrm{CB}(G)$ has the norm topology where $l_g f(x) = f(gx)$, $x \in G$. Let $\mathrm{WAP}(G)$ (respectively $\mathrm{AP}(G)$) denote the space of continuous weakly almost periodic (respectively almost periodic) functions on G i.e. all $f \in \mathrm{CB}(G)$ such that $\{l_a f; a \in G\}$ is relatively compact in the weak (resp. norm) topology of $\mathrm{CB}(G)$. Let $L_\infty(G)$ denote the Banach space of essentially bounded complex-valued functions on G with the essential supremum norm $\|\cdot\|_\infty$ as defined in **[12**, p. 141**]**. Then $\mathrm{CB}(G)$, $\mathrm{LUC}(G)$, $\mathrm{WAP}(G)$ and $\mathrm{AP}(G)$ are translation invariant subalgebras of $L_\infty(G)$ with $\mathrm{AP}(G) \subseteq \mathrm{WAP}(G) \subseteq \mathrm{LUC}(G) \subseteq \mathrm{CB}(G)$. Furthermore, $C_0(G) \cap \mathrm{AP}(G) = \{0\}$ unless G is compact, where $C_0(G)$ is the closed subalgebra of $\mathrm{CB}(G)$ consisting of all $f \in \mathrm{CB}(G)$ vanishing at infinity. Recall that an application of the Ryll-Nardzewski fixed point theorem (**[21]**) shows that $\mathrm{WAP}(G)$ has a unique invariant mean m_G i.e. m_G is a positive linear functional on $\mathrm{WAP}(G)$ of norm one and $m_G(l_a f) = m_G(r_a f) = m_G(f)$ for all $f \in \mathrm{WAP}(G)$, where