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# A $q$-ANALOGUE OF APPELL'S $F_1$ FUNCTION, ITS INTEGRAL REPRESENTATION AND TRANSFORMATIONS

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**An extension of Askey and Wilson's $q$-beta integral is evaluated as a sum of two double series. The formula is then used to find a $q$-analogue of Appell's $F_1$ function via its integral representation as well as $q$-analogues of transformations of $F_1$ to another $F_1$ and $F_3$ functions.**

**1. Introduction.** Appell's $F_1$ and $F_3$ functions are defined by the infinite series [**7, 10, 20**]

$$(1.1) \qquad F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[\alpha]_{m+n}[\beta]_m[\beta']_n}{m!n![\gamma]_{m+n}} x^m y^n$$

and

$$(1.2) \quad F_3(\alpha,\ \alpha';\ \beta, \beta';\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[\alpha, \beta]_m\ [\alpha', \beta']_n}{m!n![\gamma]_{m+n}} x^m y^n$$

subject to usual convergence restrictions, where the shifted factorials are defined by $[a]_0 = 1, [a]_m = a(a+1)\cdots(a+m-1), m = 1, 2, \ldots,$ and $[a, b]_m = [a]_m[b]_m$. The $F_1$ function is the only one of the four Appell functions that has an integral representation in terms of a single integral [**10**, 9.3(4)]

$$(1.3) \quad F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'} du,$$

where $0 <$ Re $\alpha <$ Re $\gamma$. Letting $u = 1 - v$ leaves the form of the integral in (1.3) unchanged and gives [**10**, 9.4(1)]

$$(1.4) \quad F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = (1-x)^{-\beta}(1-y)^{-\beta'}\ F_1\left(\gamma-\alpha;\ \beta,\ \beta';\ \gamma;\ \frac{x}{x-1},\ \frac{y}{y-1}\right).$$

When $\beta' = 0$, (1.4) reduces to [**10**, 2.4(1)]

$$(1.5) \qquad {}_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{-\beta}\ {}_2F_1\left(\gamma-\alpha, \beta;\ \gamma;\ \frac{x}{x-1}\right),$$