EXTENSION THEOREMS FOR FUNCTIONS OF VANISHING MEAN OSCILLATION

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A locally integrable function is said to be of vanishing mean oscillation (VMO) if its mean oscillation over cubes in \( \mathbb{R}^d \) converges to zero with the volume of the cubes. We establish necessary and sufficient conditions for a locally integrable function defined on a bounded measurable set of positive measure to be the restriction to that set of a VMO function.

1. Introduction. Let \( F \) be a locally integrable function on \( \mathbb{R}^d \) and let \( Q \) be a cube in \( \mathbb{R}^d \) with sides parallel to the axes. (We denote the set of all such cubes in \( \mathbb{R}^d \) by \( \mathcal{Q}' \).) We denote the Lebesgue measure of \( Q \) by \( |Q| \) and the length of \( Q \) by \( l(Q) \). We denote the average of \( F \) on \( Q \) by \( F_Q \); that is \( F_Q = \frac{1}{|Q|} \int_Q F \, dt \). We say \( F \) is of bounded mean oscillation (abbreviated BMO\((\mathbb{R}^d)\) or simply BMO) if

\[
\sup_{Q \in \mathcal{Q}'} \frac{1}{|Q|} \int_Q |F - F_Q| < \infty.
\]

We denote this supremum by \( \|F\|_* \). \( \|\|_* \) defines a norm on BMO and BMO is a Banach space with respect to this norm. (We identify functions which differ by a constant.) If in (1.1) we restrict the cubes to be dyadic we obtain the space dyadic-BMO and we denote the corresponding norm by \( \|\|_{*,d} \). (By a dyadic cube we mean a cube of the form \( Q = \{k_j < x_j < (k_j + 1)2^{-n}; 1 \leq j \leq d\} \) where \( n \) and \( k_j, 1 \leq j \leq d, \) are integers.) We will denote the set of dyadic cubes of length \( 2^{-n} \) by \( D_n \) and \( Q_0 \) will denote the dyadic unit cube. The function space BMO was introduced in 1961 by John and Nirenberg [7] who proved the following fundamental theorem:

**Theorem 1.1.** Let \( F \) be a locally integrable function on \( \mathbb{R}^d \), and for each \( n \in \mathbb{Z} \) define:

\[
\overline{\mu}_n(F) = \inf \left\{ \frac{1}{h} : \sup_{l(Q) \leq 2^{-n}} \inf_{a \in \mathbb{R}} \frac{1}{|Q|} \int_Q e^{h|F-a|} < 2 \right\}.
\]

Then,

1. \( F \in \text{BMO} \) if and only if
2. \( \sup_{n \in \mathbb{Z}} \overline{\mu}_n(F) < \infty. \)