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# A PRETENDER TO THE TITLE "CANONICAL MOEBIUS STRIP"

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**The Moebius Strip that results from identifying two opposite sides of a rectangle is embedded analytically and isometrically in Euclidean 3-space, as part of the rectifying developable of the algebraic curve given in paramatric form by**

$$x = \sin t, \qquad y = (1 - \cos t)^3, \qquad z = \sin(1 - \cos t)$$

**or equivalently, by**

$$y^2 + 6x^2y - 8y + x^6 = x^3y - z^3 = 0.$$

**1.** A Moebius strip is the topological space obtained from a closed rectangle by identifying two opposite sides "with a twist", that is, so that each vertex is identified with its diagonal opposite. There are many ways to embed this space in Euclidean 3-space. One embedding that comes to mind naturally is obtained by choosing an interval on the positive half of the $x$-axis, rotating it around the $z$-axis at some fixed rate, and, at the same time, rotating it at half that rate around its perpendicular bisector in the $x$-$y$-plane. If the parameter $t$ is defined as the angle through which the first rotation has gone, and $s$ is length measured along the rotating interval, the equations of the surface are easily seen to be analytic functions of the parameters.

In the real world, Moebius Strips are made out of paper: a paper rectangle whose length is sufficiently large compared to its width can be made into a Moebius Strip, without doing violence to the paper (such as tearing, stretching or creasing it), by bending it smoothly, and pasting together two edges in the appropriate manner. Is the first embedding an acceptable mathematical model for the "real" strip? The answer is "no". Our qualifications of what may be done to the paper amount to requiring the embedding to be an isometry of the metric space defined by identifying two opposites of a rectangle appropriately. An isometry would preserve the Gaussian curvature, which is 0 for the rectangle, while the surface obtained in the analytical embedding above is easily seen to have everywhere negative curvature.

Recently Carmen Chicone [**1**] has shown that there exist analytic embeddings of the Moebius Strip as a regular flat surface in 3-space. A