

WEAKLY ALMOST PERIODIC SEMIGROUPS OF OPERATORS

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We address the question as to when a motion or almost-orbit u of a strongly continuous semigroup $(S(t))_{t \geq 0}$ of operators in a Banach space X will be weakly almost periodic in the sense of Eberlein. In particular, we show (a) that this is the case in practice exactly when u uniquely decomposes as the sum $u = S(\cdot)y + \varphi$ of an almost periodic motion $S(\cdot)y: \mathbb{R}^+ \rightarrow X$ of $(S(t))_{t \geq 0}$ and a function $\varphi: \mathbb{R}^+ \rightarrow X$ that vanishes at infinity in a certain weak sense, and (b) that an almost-orbit u of a uniformly bounded C_0 -semigroup of linear operators will be weakly almost periodic provided only that u has weakly relatively compact range. Our results on existence and representation are then applied to a qualitative study of asymptotic behavior of solutions to the abstract Cauchy problem in which the focus is on almost periodicity properties and ergodic theorems.

Introduction. The point of departure for our work in this paper is the problem of distinguishing properties related to almost periodicity for solutions to the abstract Cauchy problem associated with the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$ of operators in a Banach space X . In the homogeneous case, solutions can be realized as motions of $(S(t))_{t \geq 0}$, and so the study of their asymptotic behavior reduces to a corresponding study of semigroup motions. Since weak or integral solutions to the inhomogeneous problem are almost-orbits of the associated semigroup in many concrete instances, moreover, we include almost-orbits as well as motions of $(S(t))_{t \geq 0}$ within the scope of our investigation.

In [21] (and the survey article [22]), we have treated the problem of characterizing motions that are asymptotically almost periodic; i.e., those for which their set of translates is relatively compact in the sup-normed space $(C_b(\mathbb{R}^+, X), \|\cdot\|_\infty)$ of all bounded continuous functions from \mathbb{R}^+ into X . Our present purpose is a study of motions and almost-orbits that are weakly almost periodic in the sense of Eberlein [9], which corresponds to the set of translates being relatively compact with respect to the weak topology of the Banach space $(C_b(\mathbb{R}^+, X), \|\cdot\|_\infty)$. Following a preliminary section, we show in §2 that (a) an almost-orbit $u: \mathbb{R}^+ \rightarrow X$ of $(S(t))_{t \geq 0}$ is weakly almost pe-