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CODES, TRANSFORMS AND THE SPECTRUM OF THE SYMMETRIC GROUP

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Let G_n be the graph of permutations with edges drawn between permutations differing by an adjacent transposition. Using the Kazhdan-Lusztig representations of S_n and combinatorial arguments, we show that integers frequently occur in the spectrum of G_n. That 0 and -1 are among the integers which arise has application to finite Radon transforms and to existence of perfect 1-codes on G_n.

1. Introduction. Let G_n denote the graph with vertices labeled by the permutations of $\{1, \ldots, n\}$ and with edges drawn between two vertices if and only if the two permutations differ by an adjacent transposition. This graph is sometimes called the "permutohedron" [**Be**]. It is also the graph of the Hasse diagram for the weak order of the symmetric group [**Bj**]. We say that a number is *an eigenvalue of* G_n if it is an eigenvalue of its adjacency matrix. Similarly we refer to the spectrum of G_n when we really mean the spectrum of the adjacency matrix.

In this paper we investigate the occurrence of integer eigenvalues for this graph. We were originally led to this investigation because of a coding problem: does there exist a perfect 1-code on G_n? That is, is there a collection of vertices C of G_n such that the sets $P_v = \{w | w = v \text{ or } w \text{ adjacent to } v\}$, for each $v \in C$, partition all the vertices of G_n? Such a code exists for $n = 3$ (see Figure 1), but not for any other $n < 12$. We will show that if such a code exists, then -1 must be an eigenvalue of G_n. We will also show that -1 is always an eigenvalue of G_n.

Questions about the spectrum of G_n also arise in other settings. In a later section we will define a finite Radon transform on S_n whose invertibility is dependent on the existence (or non-existence) of certain eigenvalues of G_n. The spectrum of G_n is also relevant in analyzing the behavior of certain shuffling problems [**DS**]. While most of these applications involve knowing only whether 0 or 1 is in the spectrum, the more general problem of what integers appear in the spectrum leads to better techniques in resolving these specific cases and is a more