PACIFIC JOURNAL OF MATHEMATICS
Vol. 143, No. 1, 1990

NOTE ON THE INEQUALITY OF THE ARITHMETIC AND GEOMETRIC MEANS

HAO ZHI CHUAN

We show how to insert a continuum of additional terms (defined by an integral and depending on an arbitrary positive parameter) between the two sides of the generalized arithmetic-geometric mean inequality with weights. Applications give an inequality involving positive definite matrices and also a refinement of the inequality connecting the inscribed and circumscribed radii of a triangle.

We suppose throughout that

(1) $n \in \mathbb{N}$ and $a_j > 0, \quad q_j > 0\ (j = 1, \ldots, n), \quad q_1 + \cdots + q_n = 1.$

Then we have the well-known inequality of the means (e.g. [2, #9])

$$\prod_{j=1}^{n} a_j^{q_j} \le \sum_{j=1}^{n} q_j a_j, \tag{2}$$

with equality if and only if $a_j = a_1 \quad (j = 1, \ldots, n)$.

THEOREM 1. *If* (1) *holds and if* $p > 0$, *then*

$$\prod_{j=1}^{n} a_j^{q_j} \le \left\{ p \int_0^\infty \left[\prod_{j=1}^{n} (x + a_j)^{q_j} \right]^{-p-1} dx \right\}^{-1/p} \le \sum_{j=1}^{n} q_j a_j. \tag{3}$$

Proof. For $x \ge 0$, we replace a_j by $x + a_j$ in (2); then

$$0 < \prod_{j=1}^{n} (x + a_j)^{q_j} \le \sum_{j=1}^{n} q_j (x + a_j) = x + \sum_{j=1}^{n} q_j a_j.$$

Hence (for $p > 0$)

$$\int_0^\infty \left[\prod_{j=1}^{n} (x + a_j)^{q_j} \right]^{-p-1} dx \ge \int_0^\infty \left[x + \sum_{j=1}^{n} q_j a_j \right]^{-p-1} dx = \frac{1}{p} \left(\sum_{j=1}^{n} q_j a_j \right)^{-p}. \tag{4}$$