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FIXED POINTS FOR ORIENTATION PRESERVING HOMEOMORPHISMS OF THE PLANE WHICH INTERCHANGE TWO POINTS

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Suppose h is an orientation preserving homeomorphism of the plane which interchanges two points p and q. If A is an arc from p to q, then h has a fixed point in one of the bounded complementary domains of $A \cup h(A)$.

1. Introduction. Brouwer's Lemma [2], one version of which is that each orientation preserving homeomorphism of the plane with a periodic point has a fixed point, has had much attention in the last few years. It has played a central role in some work of Fathi [7], Franks [**8**, **9**], Pelikan and Slaminka [**11**], Slaminka [**12**] and the author [**3**, **4**].

An interesting special case is when the periodic point has period two. Indeed, this case is at the heart of Fathi's argument in [7], and his proof of Brouwer's lemma requires a separate proof of this case. The purpose of this note is to show that this result follows from a particularly simple and elegant application of the notion of index of a homeomorphism along an arc. Furthermore, we get constructive information about the location of the fixed point. Our proof both simplifies and strengthens a result of Galliardo and Kottman [**10**].

In a final section we illustrate some techniques which can be used to locate fixed points more precisely.

2. The index. Let f, g be maps of the interval [01] into the plane such that $f(t)$ is distinct from $g(t)$ for each t in [01]. Then index (f, g) is defined to be the total winding number of the vector $g(t)-f(t)$ as t runs from 0 to 1. For example, in Figure 1 this vector makes a total of 1 and 1/2 turns in the clockwise (i.e., negative) direction, so the index is $-(1+1/2)$. The reader who wishes a more precise definition of index and its properties should consult [**5**] and [**6**].

If f and f' are two maps of [01] into the plane such that $f(1) = f'(0)$ then we denote by $f * f'$ the map of [01] into the plane which is $f(2t)$ on $0 \leq t \leq 1/2$, and $f'(2t-1)$ on $1/2 \leq t \leq 1$. Clearly, if index(f, g) and index(f', g') are defined then index$(f * f', g * g')$ is well defined and equal to index$(f, g)+$ index(f', g').