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ON THE FIX-POINTS OF COMPOSITE FUNCTIONS

WALTER BERGWEILER

Gross has conjectured that a composite transcendental entire function has infinitely many fix-points. We show that the conjecture is true if one of the two components has finite order.

1. Introduction and results. Let f and g be two nonlinear entire functions, at least one of them transcendental. Gross [**4**] has conjectured that the composite function $f \circ g$ has infinitely many fix-points.

Gross and Osgood [**5**] have proved that the conjecture is true, if one of the functions f and g is of finite order while the other one is of finite lower order. The conjecture has also been proved under various other conditions on f and g (cf. [**6**], [**9**], [**13**], [**14**]).

We shall prove

THEOREM 1. *Let f and g be nonlinear entire functions, at least one of them transcendental. If one of the functions f and g is of finite order, then $f \circ g$ has infinitely many fix-points.*

As a consequence of Theorem 1 we obtain

THEOREM 2. *Let f and g be nonlinear entire functions, at least one of them transcendental. If*

$$\limsup_{r \to \infty} \frac{\log\log\log M(r, f \circ g)}{\log r} < \infty,$$

then $f \circ g$ has infinitely many fix-points.

These two theorems contain and generalize many of the results referred to above.

2. Lemmas. Our proofs will be based partially on Nevanlinna theory (for notations see [7]), but mainly on Wiman-Valiron theory. We denote the maximum term of an entire function h by $\mu(r, h)$ and the central index by $N = N(r, h)$. By F we denote an exceptional set of finite logarithmic measure, not necessarily the same at each occurrence. For the convenience of the reader we state the results of