MASS OF RAYS ON COMPLETE OPEN SURFACES

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The total curvature of a complete open surface describes certain properties of the Riemannian structure which defines it. We study relationships between the total curvature and the mass of rays on a finitely connected complete open surface and obtain some integral formulas.

0. Introduction. Throughout this paper let $M$ be a connected, finitely connected, oriented, complete and noncompact Riemannian 2-manifold without boundary. The total curvature $c(M)$ of $M$ is defined to be an improper integral over $M$ of Gaussian curvature $G$ with respect to the area element $dM$ of $M$. A well-known theorem due to Cohn-Vossen [1] states that if $M$ admits total curvature, then $2\pi \chi(M) - c(M) \geq 0$, where $\chi(M)$ is the Euler characteristic of $M$. Clearly $c(M)$ depends on the choice of Riemannian metric. This phenomenon gives rise to the idea that the value $2\pi \chi(M) - c(M)$ should describe certain properties of Riemannian metric which defines it.

A ray (respectively, a straight line) on $M$ is by definition a unit speed geodesic parametrized on $[0, \infty)$ (respectively, on $\mathbb{R}$) every subarc of which realizes distance between its terminal points. For a point $p \in M$ let $S_p(1)$ be the unit circle centered at the origin of the tangent space $M_p$ to $M$ at $p$. Let $A(p)$ be the set of all unit vectors tangent to rays emanating from $p$. $A(p)$ is closed in $S_p(1)$. Let $\mathfrak{m}$ be the natural measure on $S_p(1)$ induced from the Riemannian metric. A relation between the mass of rays and the total curvature was first investigated by Maeda in [6], [7]. He proved that if $M$ is homeomorphic to $\mathbb{R}^2$ and if $G \geq 0$, then $\mathfrak{m} \circ A \geq 2\pi - c(M)$, and in particular $\inf_M \mathfrak{m} \circ A = 2\pi - c(M)$. These results were extended by Shiga in [10], [11] to Riemannian planes whose Gaussian curvatures change sign, and later by Oguchi [9] to finitely connected $M$ with one endpoint. In connection with an isoperimetric problem discussed by Fiala [3] and Hartman [4], the first-named author proved in [14] that if $M$ has one end and if $2\pi \chi(M) - c(M) < 2\pi$, then for every monotone increasing sequence $\{K_j\}$ of compact sets with $\bigcup K_j = M$, 349