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A SHORT PROOF OF ISBELL'S ZIGZAG THEOREM

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Isbell's Zigzag Theorem, which characterizes semigroup dominions (defined below) by means of equations, has several proofs. We give a short proof of the theorem from first principles.

The original proof Isbell [4] and that of Philip [6] are topological in flavour. The algebraic proofs of Howie [2] and Storrer [8] are based on work by Stenstrom [7] on tensor products of monoids. Yet another proof, using the geometric approach of regular diagrams, is due to David Jackson [5]. This latter approach also employs HNN extensions of semigroups to solve the problem. In this note we follow Jackson's lead in using what is essentially a HNN extension for our embedding (instead of the more intractable free product with amalgamation) to derive a short and direct proof of the Zigzag Theorem.

Following Howie and Isbell [3] we say that a subsemigroup U of a semigroup S *dominates* an element $d \in S$ if for every semigroup T and all morphisms $\phi_1: S \to T$, $\phi_2: S \to T$, $\phi_1|U = \phi_2|U$ implies that $d\phi_1 = d\phi_2$. The set of all elements in S dominated by U is called the *dominion* of U in S; it is obviously a subsemigroup of S containing U, and we denote it by $\mathrm{Dom}(U, S)$. Dominions are connected with epimorphisms (pre-cancellable morphisms) by the fact that a morphism $\alpha: S \to T$ is epi iff $\mathrm{Dom}(S\alpha, T) = T$.

ISBELL'S ZIGZAG THEOREM. *Let U be a subsemigroup of S. Then $d \in \mathrm{Dom}(U, S)$ if and only if $d \in U$ or there exists a sequence of factorizations of d as follows:*

$$d = u_0y_1 = x_1u_1y_1 = x_1u_2y_2 = x_2u_3y_2 = \cdots = x_mu_{2m-1}y_m = x_mu_{2m},$$

where

$$u_i \in U, \quad x_i, y_i \in S, \quad u_0 = x_1u_1, \quad u_{2i-1}y_i = u_{2i}y_{i+1},$$
$$x_iu_{2i} = x_{i+1}u_{2i+1} \quad (1 \le i \le m-1) \quad \textit{and} \quad u_{2m-1}y_m = u_{2m}.$$

Such equations are known as a *zigzag* in S over U with *value* d, *length* m, and *spine* the list $u_0, u_1, \ldots, u_{2m}$. For a survey on