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# SPECTRAL SYMMETRY OF THE DIRAC OPERATOR FOR COMPACT AND NONCOMPACT SYMMETRIC PAIRS

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**The aim of this paper is to prove a vanishing of theorem for the Dirac operator on a symmetric pair. In fact, we prove a stronger result: that the Dirac operator has spectral $G$-symmetry.**

THEOREM 1.1. *Let $(G, K)$ be a symmetric pair of rank two or greater, of compact or noncompact type and $\Gamma \subset G$ a co-compact discrete subgroup. Let $\rho$ be a metric on $\Gamma\backslash G$ whose lift to $G$ is $G$-left and $K$-right invariant. Then, the Dirac operator has spectral $G$-symmetry: that is, for each eigenvalue $\lambda$ the eigenspace $V_\lambda$ is $G$-isomorphic to the eigenspace $V_{-\lambda}$.*

COROLLARY 1.2. *The equivariant $\eta$-function vanishes identically: $\eta_G(s, g) = 0$.*

The importance of the eta invariant and questions of spectral symmetry has long been recognized, see [**1**]. If $\dim G \neq 4k+3$, the spectrum is symmetric for algebraic reasons. However, as the example in [**4**] shows, this spectrum need not be symmetric if $\dim G = 3$. For an odd dimensional simply connected Lie group with bi-invariant metric, the map $x \mapsto x^{-1}$ is an orientation reversing isometry and we again get spectral symmetry. However, this map may well not descend to quotients $\Gamma\backslash G$; for example, we know the spectrum for $SO(3) \cong SU(2)/\{\pm 1\}$ is not symmetric. Furthermore, if $G$ is a noncompact rank one group and $\Gamma$ a co-compact discrete subgroup then, with respect to certain natural metrics on $\Gamma\backslash G$, the spectrum fails to be symmetric, see [**6**]. Thus, the result does not hold in the rank one case.

In §2 we discuss the case of a symmetric pair of compact type. This is done in some detail. Section 3 contains the case of noncompact type. Since this is similar to the compact type, we concentrate on presenting the changes in the new case. We do not consider the case of a symmetric pair of Euclidean type.