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ON MINIMAL AND MAXIMAL EIGENVALUE GAPS AND THEIR CAUSES

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We consider quantum-mechanical potentials giving rise to minimal (or maximal) eigenvalue gaps subject to L^p constraints in n-dimensions. We prove existence and characterization theorems for optimizing potentials. The tunneling effect through a single barrier is shown always to be the cause of minimal gaps, and in some cases the gap minimizers are shown to be specific double-well potentials.

I. Introduction. Let Ω be a bounded, smooth domain in $\mathscr{R}^n$, and consider the Schrödinger operator

$$H = -\Delta + V(x)$$

acting on $L^2(\Omega)$, with zero Dirichlet boundary conditions. As is well known, for reasonable potentials V the spectrum consists of eigenvalues $\{E_i\}$, conventionally numbered in an increasing sequence,

$$-\infty < E_1 < E_2 \leq E_3 \leq \cdots . \tag{1.1}$$

The eigenvalues correspond to the energy levels, in atomic units, of a quantum particle in the potential energy V, imagined as $+\infty$ outside Ω. We refer to E_1 as the ground state, E_2 as the first excited state, and

$$\Gamma = E_2 - E_1 \tag{1.2}$$

as the fundamental gap.

Bounds on the fundamental gap have been the subject of a number of recent works **[1]**, **[23]**, **[25]**, usually with assumptions imposed on both V and Ω that can loosely be characterized as convex. If R is a characteristic diameter of the problem, then with these assumptions Γ can be no smaller than const. R^{-2}. Without the convex assumptions, on the other hand, exponentially small fundamental gaps $\Gamma = O(\exp(-\text{const.}\ R))$, are known to arise in double-well problems, owing to the tunneling effect, and also in problems on pinched or dumbell-shaped domains **[4]**, **[19]**. Recently, a pair of papers by Kirsch and Simon **[14]**, **[15]** established, roughly, that the fundamental gap is bounded below by a polynomial in R times $\exp(-\text{const.}\ R)$,