PACIFIC JOURNAL OF MATHEMATICS
Vol. 147, No. 2, 1991

ON TWO POLYNOMIAL SPACES ASSOCIATED WITH A BOX SPLINE

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The polynomial space $\mathscr{H}$ spanned by the integer translates of a box spline M admits a well-known characterization as the joint kernel of a set of homogeneous differential operators with constant coefficients. The dual space $\mathscr{H}^*$ has a convenient representation by a polynomial space $\mathscr{P}$, explicitly known, which plays an important role in box spline theory as well as in multivariate polynomial interpolation.

In this paper we characterize the dual space $\mathscr{P}$ as the joint kernel of simple differential operators, each one a power of a directional derivative. Various applications of this result to multivariate polynomial interpolation, multivariate splines and duality between polynomial and exponential spaces are discussed.

1. Introduction. The space $H(\phi)$ of all exponentials in the linear span $S(\phi)$ of the integer translates of a compactly supported distribution ϕ is of basic importance in multivariate spline theory since, in principle, it allows the construction of good approximation maps to $S(\phi)$ from spaces containing $S(\phi)$. Generically, $H(\phi)$ is D-invariant (i.e., closed under differentiation), hence is the joint kernel for a set $\{p(D): p \in I_{H(\phi)}\}$ of differential operators with constant coefficients, with $I_{H(\phi)}$ a polynomial ideal of finite codimension (in the space Π of all multivariate polynomials, i.e., an ideal of transcendental dimension 0, hence with finite variety). An understanding of the interplay between the space $H(\phi)$ and its associated ideal $I_{H(\phi)}$ is useful in the determination of the basic properties of $H(\phi)$ such as its spectrum, its dimension, and its local approximation order.

For the important special case when ϕ is a polynomial box spline (and $\mathscr{H} := H(\phi)$ is thus a polynomial space), an explicit set of generators for the ideal $I_{\mathscr{H}}$ is known **[BH1]**, but nevertheless, the construction of their joint kernel was found to be very difficult. At the same time, a polynomial space $\mathscr{P}$ (of very simple structure) is known which serves as a natural dual for $\mathscr{H}$ and is of substantial use in the analysis of $\mathscr{H}$. Specifically, the duality between $\mathscr{H}$ and $\mathscr{P}$ has been used in **[DR1]** in the investigation of the local approximation order of some exponential spaces, in **[DR1,2]** in the solution of an