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ON A CONSTRAINED VARIATIONAL PROBLEM AND THE SPACES OF HORIZONTAL PATHS

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In this paper we study geometry associated to an isoholonomy variational problem on a fat bundle. We prove that the energy function satisfies the Palais-Smale condition on the space of horizontal paths. We blow up the singularity of the horizontal loop space. Then we study the closed geodesics. We relate the number of connected components of the space of loops with trivial holonomy to the topology of the fat bundle.

1. Introduction. In this paper we will study geometry associated to the following constrained variational problem:

Problem 1. Consider a principal G-bundle (G compact) over a compact Riemannian manifold M

$$\pi: F \to M$$

with a given connection form ω on F. Fixing two points $x_0, x_1 \in F$, consider the space of H^1-horizontal paths from x_0 to x_1, denoted by $\Omega F(x_0, x_1)$. Then the problem is to find a horizontal path $\gamma_0 \in \Omega F(x_0, x_1)$ such that

$$E(\gamma_0) = \min_{\gamma \in \Omega F(x_0, x_1)} E(\gamma),$$

where $E(\gamma) = \int_0^1 \|\pi_* \gamma(t)\|^2\, dt$. Recall that a path γ in F is horizontal if

$$\omega(\dot{\gamma}) = 0. \tag{1.1}$$

The most important case in application (cf. [**18**]) is when the two end points lie on the same fiber, and the problem reduces to the following form:

Problem 2. Let $\Omega M(x_0, H_0)$ be the space of H^1-loops on M based at $x_0 \in M$ with holonomy H_0, then the problem is to find a loop γ_0 such that

$$E(\gamma_0) = \min_{\gamma \in \Omega M(x_0, H_0)} E(\gamma),$$

where $E(\gamma) = \int_0^1 \|\dot{\gamma}\|^2\, dt$.