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INVARIANTS FOR 3-MANIFOLDS FROM THE COMBINATORICS OF THE JONES POLYNOMIAL

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The bracket polynomial of Kauffman first gave an exceedingly simple definition of the Jones polynomial for links. Here it is used to give a short direct proof of the existence of a few of Witten's 3-manifold invariants.

The techniques of quantum field theory have been used by Witten [9] in the production of an array of invariants for 3-manifolds and for links in 3-manifolds. When the 3-manifold is the 3-sphere, these link invariants become the Jones polynomial (or one of its generalisations) evaluated at various complex roots of unity. A proof of the existence of such invariants has been given by Reshetikhin and Turaev [8] using deep results from the theory of quantum groups. An alternative approach, based on only the general outline of their method, is given here. This proof of the invariants' existence uses nothing but simple combinatorics and the well known theory of 3-manifolds being created by surgery on the 3-sphere. The result actually obtained here estiblishes only a very small selection of the new invariants, but the method has scope for extension (see however Appendix 2). The nature of the invariants is described in a fairly simple way, and those invariants that are here established are the only ones for which calculation seems to be at all feasible. Some of these calculations have been performed and discussed by Kirby and Melvin [4].

The basic tool that will be used is the bracket polynomial invariant of Kauffman [2], [7]. The bracket is a function

$\langle\ \rangle$: {Diagrams in $\mathbf{R}^2 \cup \infty$ of unoriented links} $\to \mathbf{Z}[A^{\pm 1}]$ that is defined by three properties:

(i) $\langle \varnothing \rangle = 1$;

(ii) $\langle D \cup U \rangle = \delta \langle D \rangle$, where U is a component with no crossing at all and $\delta = -A^{-2} - A^2$;

(iii) $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle \supset\subset \rangle$, where this refers to three diagrams identical except where shown.

(Note that the normalisation of (i) is not entirely standard.) It is