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TORUS ORBITS IN G/P

HERMANN FLASCHKA AND LUC HAINE

Let G be a complex semisimple Lie group of rank l, with fixed Borel subgroup B and maximal torus H. Let P be a standard parabolic subgroup. The torus H acts on G/P by $gP \mapsto hgP$. The closure X in G/P of an orbit $\{hgP|h \in H\}$ is called a *torus orbit* if it is l-dimensional and satisfies a certain genericity condition; it is a rational algebraic variety whose structure is intimately related to Lie theory, symplectic geometry, and the theory of convex bodies. This paper presents: (1) an abstract description of the torus orbit X by means of a rational polyhedral fan; (2) a description of the torus-invariant divisor whose linear system provides a natural embedding (the *Plücker embedding*) of X into a projective space; (3) a discussion of the correspondence between this divisor and the momentum mapping associated to the action on X of the compact torus $T \subset H$; (4) a list of generators of the ideal defining the Plücker embedding; (5) a formula for the intersection multiplicity of certain important torus invariant divisors on X.

We have encountered torus orbits in several problems, and the calculations just mentioned have proved useful in those other studies. In work with N. Ercolani, we find torus orbits as compactified (complex) level varieties in a certain integrable Hamiltonian system, the so-called Toda lattice. A. Bloch, T. Ratiu, and Flaschka use torus orbits in the compact setting, K/T rather than G/B, to prove a convexity theorem for a "Hermitian" Toda lattice (Duke Math. J., to appear). In collaboration with R. Cushman, we study Gröbner bases for projective embeddings of torus orbits; these are simpler than, but in some model cases dual to, the standard monomials on G/P itself. Finally, the theory of integrable systems suggests that a detailed understanding of torus orbits in loop groups might be useful and interesting.

Because the summary of necessary definitions from the theory of toric varieties takes several pages (cf. §2), we devote this Introduction mostly to a description of results that can be stated without much specialized apparatus. Just a few words about items (1), (2), and (3) above. In §3, we establish some properties of the image of the momentum map referred to in point (3); it is a convex polytope with vertices in the weight lattice. The fan Δ defining X as toric variety is