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A REMARK ON THE SYMMETRY OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS

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This note gives a necessary and sufficient condition for solutions of second order elliptic equations to be radially symmetric.

1. Introduction.

1.1. In an elegant paper **[GNN]**, Gidas-Ni-Nirenberg proved that the positive solutions of

$$\begin{cases} \Delta u = f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u \in C^2(\overline{B}), \end{cases} \tag{1}$$

must be radially symmetric. Here f is C^1 and B is the n-dimensional ball: $\{x \in R^n; |x| < 1\}$. Obviously a symmetric solution of (1) is not necessary to be positive. In this note, we give a necessary and sufficient condition for symmetric solutions of (1). The main result is the following

THEOREM 1. *Suppose $n \geq 2$. A solution u of (1) is radially symmetric if and only if its nodal set $\{x \in \overline{B}; u(x) = 0\}$ is radially symmetric.*

REMARK. It is interesting to note that Theorem 1 need not hold in case $n = 1$. For, $u = \sin x$ solves

$$u'' = -u \quad \text{in } [-\pi, \pi]$$

with the symmetric nodal set $\{0\} \cup \{-\pi, \pi\}$, but u is not radially symmetric since $\sin(-x) = -\sin x$.

It is clear that the result of **[GNN]** is a special case of Theorem 1 since the nodal set of a positive solution to (1) is the sphere ∂B.

In order to prove Theorem 1, we need the following two preliminary results.

THEOREM 2. *Let $u \in C^2(\overline{B})$ satisfy*

$$\Delta u = f(u) \quad \textit{in } B. \tag{2}$$