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ON THE HEAT EQUATION FOR HARMONIC MAPS FROM NON-COMPACT MANIFOLDS

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Harmonic maps are critical points of the energy functional for maps between Riemannian manifolds. In this paper we study the heat equation for harmonic maps from a non-compact manifold M into N. We show that if the target manifold N is compact and has non-positive sectional curvature, and if the initial map has finite total energy, then there exists a solution $u(x, t)$: $M \times [0, \infty) \to N$ and a sequence $t_j \to \infty$, such that $u(\cdot, t_j)$ converges on compact subsets of M to a harmonic from M into N. We also obtain some basic properties of the solution $u(x, t)$. In particular, we prove a uniqueness theorem for the solution and a monotonicity theorem for the energy functional.

Eells and Sampson proved that if (M, g) and (N, g') are compact Riemannian manifolds, (N, g') has non-positive sectional curvature, then any smooth map $h : M \to N$ is homotopic to a smooth harmonic map. They established the existence of a solution $u(x, t)$: $M \times [0, \infty) \to N$, of (1.1) in §1, and showed that there exists $t_j \to \infty$, such that $u(\cdot, t_j)$ converges to a smooth harmonic map from M into N. Schoen and Yau showed that if M is complete non-compact and if $h : M \to N$ has finite energy, then h is homotopic on any compact subsets of M to a harmonic map. Their method is based on Hamilton's results on harmonic maps from a manifold with boundary. By studying the heat equation directly, we recovered the result of Schoen and Yau. We believe the basic properties of solutions of the heat equation established in this paper will be useful in the study of harmonic maps on non-compact manifolds.

1. Existence. Let (M^m, g) and (N^n, g') be complete Riemannian manifolds. M is non-compact. We want to study the initial value problem for the heat flow for harmonic maps. More precisely, we want to study the following system for a map $u : M \times [0, \infty) \to N$, in local coordinates $x = (x^1, \dots, x^m)$, and $u = (u^1, \dots, u^n)$ on M and N respectively:

$$(1.1) \qquad \begin{cases} \Delta_M u^\alpha - \dfrac{\partial u^\alpha}{\partial t} = -g^{ij} \dfrac{\partial u^\beta}{\partial x^i} \dfrac{\partial u^\gamma}{\partial x^j} \Gamma'^\alpha_{\beta\gamma}, \\ \qquad\qquad\qquad \text{in } M \times (0, \infty), \alpha = 1, \dots, n; \\ u(x, 0) = h(x), \end{cases}$$

where Δ_M is the Laplace-Beltrami operator on M, $\Gamma'^\alpha_{\beta\gamma}$ are the