PACIFIC JOURNAL OF MATHEMATICS
Vol. 154, No. 1, 1992

STUDYING LINKS VIA CLOSED BRAIDS I: A FINITENESS THEOREM

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This paper is the first in a series which study the closed braid representatives of an oriented link type $\mathscr{L}$ in oriented 3-space. A combinatorial symbol is introduced which determines an oriented spanning surface F for a representative L of $\mathscr{L}$. The surface F is in a special position in 3-space relative to the braid axis A and the fibers in a fibration of the complement of A. The symbol simultaneously describes F as an embedded surface and L as a closed braid. Therefore it is both geometrically and algebraically meaningful. Using it, a complexity function is introduced. It is proved that $\mathscr{L}$ is described by at most finitely many combinatorial symbols, and thus by finitely many conjugacy classes in each braid group $\mathbf{B}_n$ when the complexity is minimal.

1. Introduction. This paper is the first in a series of papers in which the authors study the closed braid representatives of oriented links in oriented 3-space. Our goal is to develop a "calculus" for links in S^3. By this we mean a systematic procedure which begins with an arbitrary representative $\mathbf{L}$ of an arbitrary link type $\mathscr{L} = [\mathbf{L}]$, assigns to it an appropriate measure of complexity, detects when the complexity is not minimal and modifies $\mathbf{L}$ via a sequence of links of decreasing complexity to a canonical representative or a finite set of canonical representatives.

In this manuscript we will set up basic machinery which will be used throughout the series and will prove that our problem is solvable in the following sense: (i) we define an appropriate complexity function, and (ii) we prove that there are only finitely many conjugacy classes in B_∞ which represent a given link type and have minimum complexity. However, there is more to it than that. The complexity function which we define will be seen to be related to rich and interesting combinatorics. Our plan in subsequent papers is to put the combinatorics described here (possibly augmented by the addition of further structure) to work to detect when the complexity is non-minimal and to discover complexity-reducing changes in $\mathbf{L}$. Additional results in the program which are complete at this writing are reported on in **[B-M]** (which presents an overview of the first six papers in the program)