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QUALITATIVE BEHAVIOR OF SOLUTIONS OF ELLIPTIC FREE BOUNDARY PROBLEMS

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A general free boundary problem is investigated and the qualitative behavior of the fixed boundary is compared with that of the fixed boundary. As an illustration, consider the following situation. Let Γ^* be a given Jordan curve in $\mathfrak{R}^2$. For each Jordan curve Γ in $\mathfrak{R}^2$ which surrounds Γ^*, we let $\Omega = \Omega(\Gamma^*, \Gamma)$ be the region between Γ^* and Γ. Let Q be the second-order elliptic operator given by

$$Qu \equiv au_{xx} + 2bu_{xy} + cu_{yy} \quad \text{in } \Omega$$

where a, b, c depend on x, y, u_x, and u_y and $ac-b^2 > 0$. Consider the free boundary problem of finding a curve Γ and a function $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma) \cap C^0(\overline{\Omega})$ such that

$$Qu = 0 \quad \text{in } \Omega$$
$$u = 1 \quad \text{on } \Gamma^*$$

and, for a fixed $\lambda > 0$,

$$u = 0, \quad |\nabla u| = \lambda \quad \text{on } \Gamma,$$

where $\Omega = \Omega(\Gamma^*, \Gamma)$. Suppose Γ and u constitute a solution of this free boundary problem. Using curves of constant gradient direction, the geometry of the free boundary Γ is compared to the geometry of the fixed boundary Γ^*. In particular, Γ is shown to have a "simpler" geometry than does Γ^*.

0. Introduction. Let a, b, $c \in C^0(\mathfrak{R}^4)$ with $ac - b^2 > 0$ in $\mathfrak{R}^4$ and define Q to be the quasilinear, elliptic, second-order partial differential operator given by

$$Qu = au_{xx} + 2bu_{xy} + cu_{yy} \tag{1}$$

for $u \in C^2$, where $a = a(x, y, p, q)$, $b = b(x, y, p, q)$, $c = c(x, y, p, q)$ and $p = u_x(x, y)$, $q = u_y(x, y)$. We are interested in the following free boundary problem.

Quasilinear free boundary problem. Given Γ^* a Jordan curve in $\mathfrak{R}^2$ or a finite collection of pairwise disjoint Jordan curves in $\mathfrak{R}^2$ and a number $\lambda > 0$, find a bounded domain $\Omega \subset \mathfrak{R}^2$, a finite collection Γ