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THE ADJOINT REPRESENTATION L-FUNCTION FOR GL(n)

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Ideas underlying the proof of the "simple" trace formula are used to show the following. Let F be a global field, and $\mathbb{A}$ its ring of adeles. Let π be a cuspidal representation of $\mathrm{GL}(n, \mathbb{A})$ which has a supercuspidal component, and ω a unitary character of $\mathbb{A}^\times/F^\times$. Let s_0 be a complex number such that for every separable extension E of F of degree n, the L-function $L(s, \omega \circ \mathrm{Norm}_{E/F})$ over E vanishes at $s = s_0$ to the order $m \geq 0$. Then the product L-function $L(s, \pi \otimes \omega \times \check{\pi})$ vanishes at $s = s_0$ to the order m. This result is a reflection of the fact that the tensor product of a finite dimensional representation with its contragredient contains a copy of the trivial representation.

Let F be a global field, $\mathbb{A}$ its ring of adeles and $\mathbb{A}^\times$ its group of ideles. Denote by $\underline{G}$ the group scheme $\mathrm{GL}(n)$ over F, and put $G = \underline{G}(F)$, $\mathbb{G} = \underline{G}(\mathbb{A})$, and $Z \simeq F^\times$, $\mathbb{Z} \simeq \mathbb{A}^\times$ for the corresponding centers. Fix a unitary character ε of $\mathbb{Z}/Z$, and signify by π a cuspidal representation of $\mathbb{G}$ whose central character is ε. For almost all F-places v the component π_v of π at v is unramified and is determined by a semi-simple conjugacy class $t(\pi_v)$ in $\widehat{G} = \underline{G}(\mathbb{C})$ with eigenvalues $(z_i(\pi_v); 1 \leq i \leq n)$. Given a finite dimensional representation r of $\widehat{G}$, and a finite set V of F-places containing the archimedean places and those where π_v is ramified, one has the L-function

$$L^V(s, \pi, r) = \prod_{v \notin V} \det(I - q_v^{-s} r(t(\pi_v)))^{-1}$$

which converges absolutely in some right half plane $\mathrm{Re}(s) >> 1$. Here q_v is the cardinality of the residue field of the ring R_v of integers in the completion F_v of F at v.

In this paper we consider the representation r of $\widehat{G}$ on the (n^2-1)-dimensional space M of $n \times n$ complex matrices with trace zero, by the adjoint action $r(g)m = \mathrm{Ad}(g)m = gmg^{-1}$ $(m \in M, g \in \widehat{G})$. More generally we can introduce the representation Adj of $G \times \mathbb{C}^\times$ by $\mathrm{Adj}((g, z)) = zr(g)$, and hence for any character ω of $\mathbb{Z}/Z$ the