PACIFIC JOURNAL OF MATHEMATICS
Vol. 154, No. 2, 1992

NEW SPECTRAL CHARACTERIZATION THEOREMS FOR S^2

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Two theorems are proved: (1) Among surfaces of revolution which are diffeomorphic to S^2, the constant curvature metric is completely characterized by the multiplicities of its eigenvalues and, (2) If the square of an eigenfunction is, again, an eigenfunction then the metric is the standard metric on S^2.

0. Introduction. The Laplacian on a surface of revolution (M, g) "splits" into a sequence of ordinary differential operators $\{L_k\}$, $k \in \mathbf{Z}$, whose sequence of Green's operators is denoted by $\{\Gamma_k\}$. A nice feature of these Green's operators is that we can compute their traces exactly. We prove in §4 that for every surface of revolution diffeomorphic to S^2:

$$\operatorname{trace}(\Gamma_k) = \frac{1}{k}, \quad \text{for } k \neq 0.$$

This formula is used in §5 to prove the following, rather surprising, fact: If the multiplicities of the eigenvalues on (M, g) are the same as those of the standard sphere, then the numerical values of the eigenvalues are the same as those of the standard sphere. One can then use a well-known result of Berger [1] to prove Theorem 5.4. It is paraphrased here as follows.

THEOREM 0.1. *Among surfaces of revolution which are diffeomorphic to S^2, the constant curvature metric is completely characterized by the multiplicities of its eigenvalues.*

Other "spectral characterization" theorems of this sort can be found in the literature (see Berger [1], Brüning and Heintze [4], Cheng [5], Goldberg and Gauchman [10], Obata [14], and Patodi [15]).

The result of Cheng is interesting in that it characterizes the standard sphere with a property of eigenfunctions rather than eigenvalues. He proves: The spheres in 3-dim Euclidean space are characterized by the fact that they have three first eigenfunctions with square sum equal to a constant.