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CHAOS IN TERMS OF THE MAP $x \to \omega(x, f)$

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Let $\mathscr{K}$ be the class of compact subsets of $I = [0, 1]$, furnished with the Hausdorf metric. Let $f \in C(I, I)$. We study the map $\omega_f : I \to \mathscr{K}$ defined as $\omega_f(x) = \omega(x, f)$, the ω-limit set of x under f. This map is rarely continuous, and is always in the second Baire class. Those f for which ω_f is in the first Baire class exhibit a form of nonchaos that allows scrambled sets but not positive entropy. This class of functions can be characterized as those which have no infinite ω-limit sets with isolated points. We also discuss methods of constructing functions with zero topological entropy exhibiting infinite ω-limit sets with various properties.

Introduction. One finds a variety of definitions of the notion of chaos for self-maps of an interval in the mathematical literature. While these definitions differ they all carry the idea, in some form or other, that points arbitrarily close together can have orbits or ω-limit sets (attractors) that spread out or are far apart. The works **[D]**, **[LY]** and **[BC]**, for example, provide three such definitions.

In the present paper we address this idea directly. We furnish the family of ω-limit sets of a continuous function f with the Hausdorff metric and ask questions related to the continuity of the map ω_f : $x \to \omega(x, f)$. While one could phrase the questions in terms of the size of the set of points of continuity of ω_f we found a more cohesive development is possible if the questions are phrased in terms of the Baire class of ω_f. This allows us to obtain results concerning continuity as corollaries, to obtain a notion of chaos strictly between the notions involving scrambled sets **[LY]** and positive entropy **[BC]**, and to obtain a complete characterization in terms of the types of ω-limit set that f possesses.

In §1 we find that ω_f is rarely continuous. We obtain several characterizations of continuity for ω_f. In particular, we find that ω_f is continuous if and only if each ω-limit set for f has cardinality 1 or 2 and the union of all ω-limit sets is connected.

In §2 we obtain some general theorems relating the Baire class of ω_f to its Borel class and to certain notions of semi-continuity of ω_f as a set valued mapping. In particular, we find that ω_f is always in