

A HYBRID OF THEOREMS OF VINOGRADOV AND PIATETSKI-SHAPIRO

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It was proved by Vinogradov that every sufficiently large odd integer can be written as the sum of three primes. We show that this remains the case when the primes so utilized are restricted to an explicit thin set. One may take, for example, the “Piatetski-Shapiro primes” $p = [n^{1/\gamma}]$ with any $\gamma > 20/21$. By a similar argument it would follow that, for arbitrary θ , $0 < \theta < 1$, and suitable $\lambda = \lambda(\theta) > 0$, one may take the set of primes for which $\{p^\theta\} < p^{-\lambda}$.

1. Introduction. The ternary Goldbach problem was solved by Vinogradov [Vi] who gave an asymptotic formula for the number of representations of the (sufficiently large) odd integer N as the sum of three primes. We state this in the form, cf. [Va],

$$(1.1) \quad R(N) \stackrel{\text{def}}{=} \sum_{p_1+p_2+p_3=N} (\log p_1)(\log p_2)(\log p_3) \\
 = \frac{1}{2} \mathfrak{S}(N) N^2 + O_A \left(\frac{N^2}{\log^A N} \right)$$

valid for arbitrary $A > 0$, where $\mathfrak{S}(N)$ is the singular series

$$(1.2) \quad \mathfrak{S}(N) = \prod_{p|N} \left(1 - \frac{1}{(p-1)^2} \right) \prod_{p \nmid N} \left(1 + \frac{1}{(p-1)^3} \right).$$

Wirsing [Wi], motivated by earlier work of Erdős and Nathanson [EN] on sums of squares, considered the question of whether one could find thin subsets S of primes which were still sufficient to obtain all sufficiently large odd integers as sums of three of them. He obtained the very satisfactory answer that there exist such sets S with the property that $\sum_{p \leq x, p \in S} 1 \ll (x \log x)^{1/3}$. This result was later rediscovered by Ruzsa. Wirsing’s result, which is obviously best possible apart from the logarithmic factor, is based on probabilistic considerations and does not lead to a subset of the primes which is constructive or recognizable.

It was Wolke who suggested the problem of finding more familiar thin sets of primes which serve this purpose and he announced his results on one such construction at an Oberwolfach meeting in 1986.