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LIE ALGEBRAS OF TYPE D_4 OVER NUMBER FIELDS

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In this paper we show how to construct all central simple Lie algebras of type D_4 over an algebraic number field. The construction that we use is a special case of a modified version of a construction due to G. B. Seligman. The starting point for the construction is an 8-dimensional nonassociative algebra with involution $\mathrm{CD}(\mathscr{B}, \mu)$ that is obtained by the Cayley-Dickson doubling process from a 4-dimensional separable commutative associative algebra $\mathscr{B}$ and a nonzero scalar μ. The algebra $\mathrm{CD}(\mathscr{B}, \mu)$ is used as the coefficient algebra for a Lie algebra $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ that can be roughly described as the Lie algebra of 3×3-skew hermitian matrices with entries from $\mathrm{CD}(\mathscr{B}, \mu)$ relative to the involution $X \to \gamma^{-1}\overline{X}^t\gamma$, where γ is an invertible diagonal matrix with scalar entries. We show that any Lie algebra of type D_4 over a number field can be constructed as $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ for some choice of $\mathscr{B}$, μ and γ. We also give isomorphism conditions for two Lie algebras constructed in this way.

As background, we note that the problem of constructing all central simple Lie algebras of a given type over a field of characteristic 0 has previously been solved for types A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 5)$, G_2 and F_4 by W. Landherr, N. Jacobson, and M. L. Tomber ([**J5**, Chapter X], [**F&F**, Section 7]). Over number fields, this problem has been solved for types E_6, E_7 and E_8 by J. C. Ferrar using the 2nd Lie algebra construction of J. Tits and the Galois cohomological results of M. Kneser, G. Harder and V. I. Cernousov ([**F1**], [**F2**], [**F3**]).

Our main tool in this paper will be an associative algebra invariant $\mathscr{E}(\mathscr{L})$, which we call the Allen invariant, that can be associated to any Lie algebra $\mathscr{L}$ of type D_4 over a field of characteristic 0. $\mathscr{E}(\mathscr{L})$ was introduced for special D_4's by Jacobson [**J2**] and in general by H. P. Allen [**All1**]. Sections 2–6 of this paper are devoted to the study of the invariant $\mathscr{E}(\mathscr{L})$. The main result obtained in these sections is a characterization, using the corestriction of algebras, of the associative algebras that can arise as Allen invariants of Lie algebras of type D_4 over a number field. In §7 (and in an appendix—§12), we use the cohomological results of Harder and Kneser to prove a general isomorphism theorem for Lie algebras of type D_4 over number