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A REMARK ON LERAY'S INEQUALITY

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In this paper we recapitulate briefly the significance of Leray's inequality in his proof of the existence of stationary solutions to the Navier-Stokes equations and show that in some simple cases it is equivalent to the flux condition on the boundary value.

1. Leray's inequality. The problem about whether or not there exist stationary solutions to the Navier-Stokes equations has been an open problem despite of a lot of efforts of many mathematicians. What has been so far obtained for this equation in this respect is an existence theorem due to Leray [2] under the condition which we call "flux condition" to be explained below.

Let D be a bounded domain with C^∞ boundary Γ in $\mathbf{R}^n$ $(n \geq 2)$. The stationary Navier-Stokes equation in D is expressed as

$$(1) \qquad \begin{cases} \Delta X - (X \cdot \nabla)X - \operatorname{grad} p = F & \text{in } D, \\ \operatorname{div} X = 0 & \text{in } D, \\ X = B & \text{on } \Gamma, \end{cases}$$

where $X = (X_1, \dots, X_n)$ is the velocity vector field, p the pressure, F the exterior force and B is the boundary condition. Δ is the Laplacian, $(\Delta X)_i = \Delta X_i$, and

$$((X \cdot \nabla)Y)_i = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j} Y_i .$$

The boundary condition B cannot be given arbitrarily. As a necessary condition of the solenoidalness condition $\operatorname{div} X = 0$ and the Gauss-Stokes formula, B should satisfy the following compatibility condition

$$(2) \qquad \int_\Gamma B \cdot \vec{n}\, dS = 0,$$

where $\vec{n}$ is the unit outer normal to the boundary Γ and dS is the surface element. The problem is whether equation (1) admits a solution (X, p) under the compatibility condition (2).