PACIFIC JOURNAL OF MATHEMATICS
Vol. 157, No. 2, 1993

ASYMPTOTICALLY FREE FAMILIES OF RANDOM UNITARIES IN SYMMETRIC GROUPS

ALEXANDRU NICA

We prove that independent, Haar distributed families of random unitaries in symmetric groups are asymptotically free.

1. Introduction. In this note we prove that independent, Haar distributed families of random unitaries in symmetric groups are asymptotically free.

If G_n is a closed subgroup of the unitary group $U(n)$, then by a random unitary in G_n we understand a measurable function $f: X \to G_n$, where $(X, \mathscr{F}, P)$ is a (fixed) probability space. A random unitary in G_n has a distribution, which is a probability measure on G_n, and we can define the notion of independence for a family of random unitaries, exactly as it is done for usual real-valued random variables. A family $(f_\omega)_{\omega\in\Omega}$ of random unitaries in G_n will be called, following **[1]**, standard-independent if it is independent and if the distribution of every f_ω is the Haar measure on G_n.

Now, a family $(f_\omega)_{\omega\in\Omega}$ of random unitaries in G_n can be also viewed as a family of unitaries in the non-commutative probability space $(\mathfrak{M}_n, \tau_n)$, where $\mathfrak{M}_n$ is the unital *-algebra of measurable functions $X \to \mathrm{Mat}_n(\mathbb{C})$, having bounded entries, and τ_n is the trace-state of $\mathfrak{M}_n$ obtained by integrating the normalized trace of $\mathrm{Mat}_n(\mathbb{C})$. From this point of view, the concept analogous to independence to be considered is the property of $(f_\omega)_{\omega\in\Omega}$ of being or not being free (see **[2]**). This property can be expressed in terms of a naturally defined "non-commutative distribution" of $(f_\omega)_{\omega\in\Omega}$, which is a state on the group algebra of the free group on Ω generators.

Hence, we are in a situation when both concepts of independence and freeness can be considered. It seems to be a deep phenomenon that, as found by D. Voiculescu in **[1]** for several important examples of such situations, one can hope independence to give rise (at least in good cases) to asymptotic freeness.

In our particular framework, the problem-type reflecting this phenomenon can be stated as follows: "For every $n \geq 1$, let $(f_{n,\omega})_{\omega\in\Omega}$