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NON-UNIQUENESS OF THE METRIC IN LORENTZIAN MANIFOLDS

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This paper is concerned with the correspondence between a Lorentzian metric and its Levi-Civita connection. Although each metric determines a unique compatible symmetric connection, it is possible for more than one metric to engender the same connection. This non-uniqueness is studied for metrics of arbitrary signature and for Lorentzian metrics is shown to arise either from a de Rham-Wu decomposition or a local parallel null vector field. A key ingredient in the analysis is the construct of a submersive connection in which a connection passes to a quotient space. Finally, two examples of metrics are given, the first of which shows that the metric may be non-unique even though a null vector field exists only locally. The second example indicates that for metrics of higher signature non-uniqueness need not result from the existence of a de Rham decomposition or parallel null vector fields.

The fundamental lemma of Riemannian or pseudo-Riemannian geometry asserts that a non-degenerate metric g determines a unique compatible symmetric connection, the so-called Levi-Civita connection. Nonetheless, it is possible for more than one metric to engender the same connection. For example, suppose that ∇ is the Levi-Civita connection of a metric g on a manifold M. (By the term "metric" shall be meant a symmetric type $(0, 2)$ non-degenerate tensor of arbitrary signature, although the main concern of this paper will be with Lorentzian metrics.) Suppose, further, that K is a vector field on M which is parallel with respect to ∇, that is, $\nabla_X K$ is zero for all vector fields X on M. Denote by α the 1-form dual to K by g and by $\alpha \odot \alpha$ its symmetric square. Then provided it is non-degenerate, $g + \lambda\alpha \odot \alpha$, where $\lambda \in \mathbb{R}$, is another metric which has ∇ as its Levi-Civita connection. The main concern of this article is to give a description of all possible metrics that are compatible with the Levi-Civita connection of a Lorentz metric.

Another situation in which there is non-uniqueness in the metric description is when (M, g) admits a de Rham decomposition. In that case M is diffeomorphic to a product $M_1 \times M_2$ of manifolds and g