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CONGRUENCE PROPERTIES OF FUNCTIONS RELATED TO THE PARTITION FUNCTION

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In this paper we describe a straightforward and almost entirely elementary method for establishing congruence properties of certain functions that are related to the partition function.

For integer k define $p_k(n)$ by

$$\prod_{m=1}^{\infty}(1-x^m)^k = \sum_{n=0}^{\infty} p_k(n)x^n .$$

In particular, $p_{-1}(n)$ is $p(n)$, the partition function and $p_{24}(n-1)$ is Ramanujan's τ-function.

We are interested in congruences of the form

$$(1) \qquad p_k(np+b) \equiv 0 \pmod{p} \text{ for all } n \geq 1$$

for prime p, as typified by the partition congruences

$$(2) \qquad p(5n+4) \equiv 0 \pmod{5},$$

$$(3) \qquad p(7n+5) \equiv 0 \pmod{7}$$

and

$$(4) \qquad p(11n+6) \equiv 0 \pmod{11}$$

discovered by Ramanujan and proved in [**13**] and [**14**]. Ramanujan also conjectured that if $24b \equiv 1 \pmod{q}$ and $q = 5^\alpha 7^\beta 11^\gamma$ then $p(qn+b) \equiv 0 \pmod{q}$. He was able to supply proofs for $q = 25$, 49 in [**13**] and $q = 121$ in an unpublished manuscript [**15**]. Ramanujan's conjecture was incorrect as stated for powers of 7 and Watson [**16**] proved a modified version; if $24b \equiv 1 \pmod{5^\alpha 7^{2\beta}}$ then $p(5^\alpha 7^{2\beta} n + b) \equiv 0 \pmod{5^\alpha 7^{\beta+1}}$. Watson's proofs have been simplified by Hirschhorn and Hunt [**6**] and Garvan [**4**]. Lehner [**9**] dealt with $q = 1331$ and the proof of the conjecture was completed by Atkin [**1**].

Congruences modulo powers of 13 have been considered by Atkin and O'Brien [**2**]. A general treatment of $p_k(n)$ modulo powers of 2, 3, 5, 7 and 13 is given in Atkin [**3**], modulo powers of 11 in