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ON ORTHOMORPHISMS BETWEEN VON NEUMANN PREDUALS AND A PROBLEM OF ARAKI

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A problem of Araki concerning the characterization of orthogonality preserving positive maps between preduals of von Neumann algebras is solved in a general setting.

Introduction. In an interesting recent article, Araki [1] initiated the study of orthogonal decomposition preserving positive linear maps (o.d. homomorphisms) between preduals of von Neumann algebras. (See below for definitions.)

Let M and N be von Neumann algebras and let $\phi: M_* \to N_*$ be a linear mapping. When either M or N is of Type I, with no direct summand of Type I_2, Araki proved that ϕ is a bijective o.d. homomorphism if, and only if, $\phi^* = z\pi$ where z is a positive invertible element of the centre of M and $\pi: N \to M$ is a Jordan isomorphism.

Araki posed the problem of establishing an analogous characterization when M and N were of Type II or Type III.

Araki used delicate Radon-Nikodym methods which seem very difficult to generalize to algebras which are not of Type I. However, by adopting a different approach, we are able to show, for arbitrary von Neumann algebras M and N, that if $\phi: M_* \to N_*$ is an o.d. homomorphism then $\phi^*\pi = z\,\mathrm{id}_M$ where z is a positive central element of M and π is a Jordan $*$ homomorphism, and we obtain a characterization in these terms. If ϕ is an o.d. isomorphism, we find that z is invertible and that π is a Jordan $*$ isomorphism. This proves that Araki's characterization of o.d. isomorphisms is valid for arbitrary von Neumann algebras M and N.

1. Preliminaries. Two positive linear functionals ρ, τ in the predual M_* of a W^*-algebra M are said to be orthogonal, written $\rho \perp \tau$, if the corresponding support projections $s(\rho)$, $s(\tau)$ are orthogonal elements in the algebra M. Every hermitian functional ρ in M_* admits a unique orthogonal decomposition $\rho = \rho_+ - \rho_-$, where $\rho_+, \rho_- \in M_*^+$ and $\rho_+ \perp \rho_-$. On the other hand every hermitian element x in M has a unique orthogonal decomposition $x = x_+ = x_-$, where $x_+, x_- \geq 0$ and $x_+ \cdot x_- = 0$.