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ON A NON-LINEAR EQUATION RELATED TO THE GEOMETRY OF THE DIFFEOMORPHISM GROUP

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Let M be a compact boundaryless Riemannian manifold. We derive the equations on M which characterize asymptotic vectors on $\mathrm{Diff}_{\mathrm{vol}}(M)$. We classify those M's whose volume-preserving diffeomorphism groups admit asymptotic vectors which are represented by *harmonic* vector fields on M. We then show that these harmonic solutions can be used to construct other (typically non-harmonic) solutions.

Introduction. In this paper, we examine the extrinsic geometry of $\mathrm{Diff}_{\mathrm{vol}}(M)$ as a submanifold of the full diffeomorphism group $\mathrm{Diff}(M)$. We derive and study the equations on M which characterize asymptotic vectors on $\mathrm{Diff}_{\mathrm{vol}}(M)$. These equations on M constitute, a priori, a second order pde system.

The contents of the paper are as follows. Section 1 reviews the weak Riemannian geometry and the 'Lie group'-like structure of $\mathrm{Diff}(M)$ and $\mathrm{Diff}_{\mathrm{vol}}(M)$. Section 2 recalls the Levi-Cività connection on $\mathrm{Diff}(M)$. Section 3 discusses the induced connection on $\mathrm{Diff}_{\mathrm{vol}}(M)$ and derives the equations for asymptotic vectors. In §4, we show that the second order pde system described in §3 is equivalent to a *single* first order equation in the compact boundaryless case. We classify those Riemannian manifolds whose volume-preserving diffeomorphism groups admit asymptotic vectors which are represented by *harmonic* vector fields on M. These harmonic solutions can be used to construct other (non-harmonic) solutions. In particular, we show that any 2-dimensional manifold carries metrics such that the corresponding volume-preserving diffeomorphism group admits asymptotic (but possibly non-harmonic!) vectors. Section 5 discusses the system of equations derived in the previous section, in the setting of noncompact boundaryless 2-dimensional manifolds.

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