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VOLUMES OF TUBULAR NEIGHBOURHOODS OF REAL ALGEBRAIC VARIETIES

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This paper concerns the following problem:

Let V be an algebraic variety in n-dimensional euclidean space. For each pair of positive numbers ρ and R find an upper bound on the volume of the set of points that are within a distance of ρ from V and within a distance R from a fixed point p_0. Obtain this upper bound so that it is independent of the choice of p_0. In particular, does there exist a universal nth-degree polynomial, say $P_n(\cdot, \cdot, \cdot)$ which automatically provides an upper bound upon entering ρ, R and the degree of V?

Introduction. Among the people who have worked on this problem are Demmel, Renegar, Ocneanu and myself. The one common factor which led us to this problem was our familiarity with Smale's Bulletin article [**S**]. In it Smale obtained an upper bound for the case where V is a complex hypersurface. His method however lacks rigor and possesses a serious flaw. This flaw involves incorrectly applying Fubini's Theorem and ignoring the special points of V where the "relative curvature" is larger than $\Omega(\frac{1}{\rho})$. Since then Renegar who was a thesis student of Smale has obtained a correct solution to this special case where V is a complex algebraic hypersurface. Later on, Demmel using some elementary results from integral geometry extended Renegar's results to all complex algebraic varieties and all tubes of length less than 1. See Theorem 4.1 of Demmel [**D**]. Crucial to Demmel's result is an estimate of the r-dimensional measure of the portion of an r-dimensional algebraic variety contained in an n-ball S with radius R. More specifically Demmel used the following fact: There exists a constant c depending only on n, so that if $\mathrm{Vol}_r(\cdot)$ is the r-dimensional measure of r-manifolds, then

$$\mathrm{Vol}_r(V \cap S) \leq c \cdot \mathrm{degree}(V) \cdot R^r.$$

For the methods used in proving this result one can see Wongkew ([**W**], pp. 9–10) or Demmel ([**D**], p.19). This fact generalizes Proposition 6.3 in Renegar [**R**].