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PRIMITIVE IDEALS AND DERIVATIONS ON NON-COMMUTATIVE BANACH ALGEBRAS

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The Singer-Wermer Conjecture states that if D is a (possibly unbounded) derivation on a commutative Banach algebra then the range of D is contained in the (Jacobson) radical of the algebra. This conjecture is now known to be true. However, it is still not currently known whether or not the Singer-Wermer Conjecture on derivations extends to non-commutative Banach algebras in the following sense: if D is a (possibly unbounded) derivation then is $D(P) \subseteq P$ for all primitive ideals P of the algebra? This has become known as the *non-commutative* version of the Singer-Wermer Conjecture. We first correct an automatic continuity result in the literature concerning which (and how many) primitive ideals can fail to be invariant. Using this result together with some representation theory we prove a theorem about derivations whose second iteration annihilates some element (specifically, $D^2a = 0$ implies that Da is quasinilpotent). This theorem does *not* require commutativity of the algebra and it is easily seen to imply the Singer-Wermer Conjecture. The proof itself is done by contradiction in which the remaining case leads to a new derivation on a commutative subalgebra, and this case can be contradicted by the arguments used in the proof of the Singer-Wermer Conjecture.

1. Automatic continuity preliminaries. The fundamental work which started investigation into the ranges of derivations on Banach algebras is due to Singer and Wermer [**8**] in 1955. In this paper, the authors proved that every *bounded* derivation on a *commutative* Banach algebra mapped into the (Jacobson) radical. They also made a very insightful conjecture, namely that the assumption of boundedness was unnecessary. This became known as the Singer-Wermer Conjecture and was proved in 1987 by the author [**9**]. The arguments used in [**9**] are rather dependent on the commutativity of the Banach algebra.

In this paper we seek a proof of a result (Theorem 2.9) about derivations on (possibly non-commutative) Banach algebras which implies the Singer-Wermer Conjecture. In results of this type, a major obstacle is the discontinuity of the derivation. This was shown very early on in the reductions (which were needed for [**9**]) of the problem by Johnson [**2**] and Johnson and Sinclair [**3**] who also established a number of fundamental principles in what is now known as the theory