PACIFIC JOURNAL OF MATHEMATICS
Vol. 159, No. 1, 1993

A CONVERSE TO A THEOREM OF KOMLÓS FOR CONVEX SUBSETS OF L_1

CHRIS LENNARD

A theorem of Komlós is a subsequence version of the strong law of large numbers. It states that if $(f_n)_n$ is a sequence of norm-bounded random variables in $L_1(\mu)$, where μ is a probability measure, then there exists a subsequence $(g_k)_k$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that for all further subsequences $(h_m)_m$, the sequence of successive arithmetic means of $(h_m)_m$ converges to f almost everywhere.

In this paper we show that, conversely, if C is a convex subset of $L_1(\mu)$ satisfying the conclusion of Komlós' theorem, then C must be L_1-norm bounded.

Introduction. A version of the strong law of large numbers in probability theory states that if $(f_n)_{n=1}^{\infty}$ is a sequence of independent, scalar-valued integrable functions (random variables), on a probability measure space (Ω, Σ, μ), each having the same distribution with mean m, then

$$\frac{1}{n}\sum_{j=1}^{n} f_j \underset{n}{\rightarrow} m \quad \text{almost everywhere.}$$

In (1967) Komlós **[Ko]** showed that arbitrary sequences of integrable random variables whose absolute values have uniformly bounded expectations always have subsequences that satisfy a version of the strong law. Indeed, for all sequences $(f_n)_{n=1}^{\infty}$ in $L_1(\mu)$ with

$$\sup_n \int_\Omega |f_n|\, d\mu < \infty,$$

there exists a subsequence $(g_k)_{k=1}^{\infty}$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that all further subsequences $(h_m)_m$ of $(g_k)_k$ satisfy

$$\frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\rightarrow} f \quad \text{almost everywhere.}$$

This result became the archetype for what Chatterji **[C2]** in the early 1970s called "the subsequence principle in probability theory". This heuristic principle led Chatterji **[C1]**, **[C2]**, **[C3]** (see also Gaposhkin **[Ga]**) to find subsequence versions of the central limit theorem and