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AN APPLICATION OF HOMOGENIZATION THEORY TO HARMONIC ANALYSIS ON SOLVABLE LIE GROUPS OF POLYNOMIAL GROWTH

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Let Q be a connected solvable Lie group of polynomial growth. Let also $E_1, \dots, E_p$ be left invariant vector fields on G that satisfy Hörmander's condition and denote by $L = -(E_1^2 + \cdots + E_p^2)$ the associated sub-Laplacian and by $S(x, t)$ the ball which is centered at $x \in Q$ and it is of radius $t > 0$ with respect to the control distance associated to those vector fields. The goal of this article is to prove the following Harnack inequality: there is a constant $c > 0$ such that $|E_i u(x)| \leq ct^{-1}u(x), x \in Q, t \geq 1, 1 \leq i \leq p$, for all $u \geq 0$ such that $Lu = 0$ in $S(x, t)$. This inequality is proved by adapting some ideas from the theory of homogenization.

0. Introduction. Let Q be a connected solvable Lie group which we assume to be of polynomial growth; i.e., if dg is a left invariant Haar measure on Q and V a compact neighborhood of the identity element e of Q, then there are constants $c, d > 0$ such that

$$dg - \text{measure}(V^n) \leq cn^d, \qquad n \in \mathbb{N}.$$

Notice that the connected nilpotent Lie groups are also solvable and of polynomial growth (cf. [**5**], [**6**]).

Let us identify the Lie algebra $\mathfrak{q}$ of Q with the left invariant vector fields on Q and consider $E_1, \dots, E_p \in \mathfrak{q}$ that satisfy Hörmander's condition; i.e., together with their successive Lie brackets $[E_{i_1}, [E_{i_2}, [\dots[E_{i_{s-1}}, E_{i_s}]\dots]]]$, they generate $\mathfrak{q}$. To these vector fields there is associated, in a canonical way, a left invariant distance $d_E(\cdot, \cdot)$ on G, called control distance. This distance has the property that (cf. [**15**]) if $S_E(x, t) = \{y \in G, d_E(x, y) < t\}$, $x \in G$, $t > 0$ then there is $c \in \mathbb{N}$ such that

$$(0.1) \qquad S_E(e, n) \subseteq V^{cn}, \quad V^n \subseteq S_E(e, cn), \qquad n \in \mathbb{N}.$$

According to a classical theorem of L. Hörmander [7] the operator

$$L = -(E_1^2 + \cdots + E_p^2)$$

is hypoelliptic.

The goal of this paper is to prove the following result: