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BETWEEN THE UNITARY AND SIMILARITY ORBITS OF NORMAL OPERATORS

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D. A. Herrero has defined the $(\mathcal{U}+\mathcal{K})$-orbit of an operator T acting on a Hilbert space $\mathcal{H}$ to be $(\mathcal{U}+\mathcal{K})(T) = \{R^{-1}TR: R$ invertible of the form unitary plus compact$\}$. In this paper, we characterize the norm closure in $\mathcal{B}(\mathcal{H})$ of such an orbit in three cases: firstly, when T is normal; secondly when T is compact; and thirdly, when T is the unilateral shift. Some consequences of these characterizations are also explored.

1. Introduction. Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators acting on $\mathcal{H}$. As usual, $\mathcal{K}(\mathcal{H})$ will denote the unique two-sided ideal of compact operators. There are many interesting ways of partitioning the set $\mathcal{B}(\mathcal{H})$ into equivalence classes. We mention two in particular.

Given $T \in \mathcal{B}(\mathcal{H})$, we define the *unitary orbit* of T as $\mathcal{U}(T) = \{U^*TU: U \in \mathcal{B}(\mathcal{H})$ a unitary operator$\}$. Then an operator $A \in \mathcal{U}(T)$ if its action on $\mathcal{H}$ is geometrically identical to that of T. Equivalently, one can think of A as T itself acting on an isomorphic copy of $\mathcal{H}$.

Another much studied class is the *similarity orbit* of $T \in \mathcal{B}(\mathcal{H})$, namely $\mathcal{S}(T) = \{S^{-1}TS: S \in \mathcal{B}(\mathcal{H})$ an invertible operator$\}$. This notion of equivalence ignores the geometry of the Hilbert space, and concentrates on the underlying vector space structure.

In general, neither of these sets need be closed. This is in contrast to finite dimensional Hilbert spaces, where $\mathcal{U}(T)$ is always closed while $\mathcal{S}(T)$ is closed if and only if T is similar to a normal matrix [**Her 1**, p. 14]. It is therefore interesting to describe the norm closure of these orbits, a program which for unitary orbits was undertaken by D. W. Hadwin [**Had**], using a result of D. Voiculescu [**Voi**], and for similarity orbits was done by C. Apostol, L. Fialkow, D. Herrero, and D. Voiculescu [**AFHV**].

One can also turn one's attention to the Calkin algebra $\mathcal{A}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and consider both unitary and similarity orbits there. Indeed, one of the major results along these lines is the classification