PACIFIC JOURNAL OF MATHEMATICS
Vol. 160, No. 2, 1993

MODULI OF LINEAR DIFFERENTIAL EQUATIONS ON THE RIEMANN SPHERE WITH FIXED GALOIS GROUPS

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For fixed m and n, we consider the vector space of linear differential equations of order n whose coefficients are polynomials of degree at most m. We show that for G in a large class of linear algebraic groups, if we fix the exponents and determining factors at the singular points (but not the singular points themselves) then the set of such differential equations with this fixed data, fixed Galois group G and fixed G-module for the solution space forms a constructible set (i.e., an element of the Boolean algebra generated by the Zariski closed sets). Our class of groups includes finite groups, connected groups, and groups whose connected component of the identity is semisimple or unipotent. We give an example of a group for which this result is false and also apply this result to the inverse problem in differential Galois theory.

1. Introduction. In this paper we consider the set $\mathscr{L}(n, m)$ of homogeneous linear differential equations

$$L(y) = a_n(x)y^{(n)} + \cdots + a_0(x) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}x^j y^{(i)} \tag{1}$$

of order at most n whose coefficients are polynomials of degree at most m with complex coefficients. By identifying $L \in \mathscr{L}(n, m)$ with the vector (a_{ij}), one sees that $\mathscr{L}(n, m)$ may be identified with an affine space $\mathbb{C}^{(n+1)(m+1)}$. Let G be a linear algebraic group and V a G-module. One would like to understand the structure of $\mathscr{L}(n, m, G, V)$, the set of $L \in \mathscr{L}(n, m)$ with Galois group $\mathrm{Gal}(L)$ equal to G and having solution space $\mathrm{Soln}(L)$ isomorphic to V as a G-module. In general $\mathscr{L}(n, m, G, V)$ is not a Zariski closed subset of $\mathscr{L}(n, m)$ or even a constructible subset of $\mathscr{L}(n, m)$ (i.e. an element of the Boolean algebra generated by the Zariski closed sets). To see this consider the family of equations $L_c(y) = xy' - cy = 0$, $c \in \mathbb{C}$. The Galois group is a subgroup of $\mathbb{C}^*$, the multiplicative group of nonzero complex numbers. It equals $\mathbb{C}^*$ if and only if c is not a rational number. If $c = p/q$, $p, q \in \mathbb{Z}$, $(p, q) = 1$, then $\mathrm{Gal}(L_c) =$