PACIFIC JOURNAL OF MATHEMATICS
Vol. 160, No. 2, 1993

THE SOFT TORUS AND APPLICATIONS TO ALMOST COMMUTING MATRICES

RUY EXEL

The "Soft Torus" A_ε is defined to be the universal C^*-algebra generated by a pair of unitaries for which the commutator has norm less than or equal to ε. We show that the K-theory of A_ε is naturally isomorphic to the K-theory of the algebra of continuous functions on the two-torus although these algebras are not homotopically equivalent. This result is applied to give a new proof of the equality of certain invariants associated to almost commuting unitary matrices.

1. Introduction. The C^*-algebra $C(\mathbf{T}^2)$ of all continuous complex valued functions on the two-torus is well known to be the universal C^*-algebra generated by two commuting unitary elements.

Softening the commuting condition we define for all ε in the real interval $[0, 2]$ the "Soft Torus" A_ε to be the universal C^*-algebra generated by a pair of elements u_ε and v_ε, subject to the relations $u_\varepsilon^* u_\varepsilon = u_\varepsilon u_\varepsilon^* = 1 = v_\varepsilon^* v_\varepsilon = v_\varepsilon v_\varepsilon^*$ and $\|u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon\| \leq \varepsilon$.

Clearly if ε is taken to be zero then A_ε is nothing but an isomorphic model of the "hard torus" $C(\mathbf{T}^2)$. On the opposite extreme if $\varepsilon = 2$ then A_ε is the full C^*-algebra of the free group on two generators. The reader is referred to **[1]** for more information on the theory of C^*-algebras defined by generators and relations.

One of the main goals of this work is the computation of the K-theory groups of A_ε. It turns out that A_ε has the same K-groups as $C(\mathbf{T}^2)$ when $\varepsilon < 2$ (if $\varepsilon = 2$ it is well known that $K_0(A_\varepsilon) = \mathbf{Z}$ and $K_1(A_\varepsilon) = \mathbf{Z} \oplus \mathbf{Z}$ **[4]**). However we shall show that A_ε is not in the homotopy class of $C(\mathbf{T}^2)$.

We say that two elements u and v in a C^*-algebra are ε-almost commuting if the commutation error $\|uv - vu\|$ is less than or equal to ε.

Several authors **[3]**, **[5]**, **[8]**, **[9]**, **[10]**, **[13]**, **[17]**, **[18]** have investigated the properties of almost commuting complex matrices in what has been called, after Brown, Douglas and Fillmore's work on essentially normal operators **[2]**, "quantitative BDF theory".

Recent works by Loring, Choi and myself **[13]**, **[3]**, **[8]** have introduced invariants which can detect obstructions to the existence of