PACIFIC JOURNAL OF MATHEMATICS
Vol. 161, No. 1, 1993

STUDYING LINKS VIA CLOSED BRAIDS III: CLASSIFYING LINKS WHICH ARE CLOSED 3-BRAIDS

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A complete solution is given to the classification problem for oriented links which are closed three-braids. The Classification Theorem asserts that, up to a finite list of exceptional cases, links which can be represented by closed 3-braids are represented by a unique conjugacy class in the group of 3-braids. The exceptional cases are the expected ones (links of braid index 1 and 2) and an unexpected infinite family of invertible links, each member of which has two 3-braid axes. The two axes correspond to diagrams which are related by "braid-preserving flypes".

An algorithm is given which begins with an arbitrary closed 3-braid (or alternatively any link diagram with 3 Seifert circles), and converts it into a normal form which characterizes its oriented link type in oriented 3-space. One can decide from the normal form whether the link is prime or composite, split or irreducible, amphicheiral and or invertible. One can decide if the braid index is 3, 2 or 1. Using related results of P. J. Xu, one may determine the genus and construct a surface of maximum Euler characteristic with boundary the given link.

It is proved that the stabilization index of a link which is represented by a closed 3-braid is ≤ 1, i.e. any two 3-braid representatives of the same link type become conjugate after a single stabilization to B_4.

1. Introduction. This manuscript is the third in a series of six papers in which the authors have been studying the closed braid representatives of links. The other five are referenced here as [**B-M I, II, IV–VI**] In [**B-M, I**] the authors prove that, with an appropriate definition of the complexity of a closed braid representative of a link, there are finitely many conjugacy classes of braids of minimum complexity. In this paper we work out in detail the first non-trivial example of this "finiteness theorem": we prove that for links of braid index $n \leq 3$ there are at most two conjugacy classes of minimum complexity. We then use this result to give a complete numerical invariant of link type, and an algorithm to find it, starting with an arbitrary closed 3-braid.

The complexity function which was used in [**B-M, I**] relates to incompressible surfaces which are bounded by the link. These surfaces have maximum Euler characteristic, among all oriented surfaces with