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TANGENTIAL AND NORMAL EULER NUMBERS, COMPLEX POINTS, AND SINGULARITIES OF PROJECTIONS FOR ORIENTED SURFACES IN FOUR-SPACE

THOMAS BANCHOFF AND FRANK FARRIS

For a compact oriented smooth surface immersed in Euclidean four-space (thought of as complex two-space), the sum of the tangential and normal Euler numbers is equal to the algebraic number of points where the tangent plane is a complex line. This follows from the construction of an explicit homology between the zero-chains of complex points and the zero-chains of singular points of projections to lines and hyperplanes representing the tangential and normal Euler classes.

1. Introduction. For a compact two-dimensional surface smoothly immersed in Euclidean four-dimensional space $\mathbf{R}^4$, also thought of as complex two-dimensional space $\mathbf{C}^2$, the sum of the Euler number of M and the Euler number of its normal bundle is the negative of the algebraic number of (isolated) *complex points* where its tangent plane is the graph of a complex linear function. Our aim in this article is to present a geometric proof of this theorem. This theorem, already implicit in the fundamental paper of Chern and Spanier **[C-S]**, has been developed and generalized in papers of Webster **[W1]**, **[W2]**, and employed as a lemma in a number of other investigations, for example the work of Hoffman and Osserman **[H-O]** and Fiedler **[F]**. These papers rely on a variety of abstract techniques from algebraic topology and differential geometry, and it is frequently not easy to see the underlying geometric content of the theorem. In this paper, we present a concrete proof of the theorem for oriented surfaces, by interpreting the Euler numbers as singularities of projections in order to obtain 0-chains representing the Poincaré duals of the tangential and normal Euler classes, and by using the Gauss map of the surface to exhibit an explicit homology between these chains and the 0-chain of (indexed) complex points.

Consider briefly the Euler numbers involved in the theorem. The (*tangential*) *Euler number* of M is the most well-known topological invariant of an oriented surface, characterized by a theorem of Heinz.