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Nonnegative bounded operators A and B on a Hilbert space $\mathscr{H}$ commute if $AB^n+B^nA \geq 0$ for $n = 1, 3, \dots,$, or if $e^{tA} \leq e^{tA+sB} \leq e^{tA+s\|B\|}$ for every $s, t > 0$.

In this paper A and B represent (not necessarily bounded) self-adjoint operators with spectral families $\{E_\lambda\}$ and $\{F_\lambda\}$, respectively, on a Hilbert space $\mathscr{H}$. We study some conditions which imply that A and B commute.

1. In general, $AB + BA$ is not necessarily nonnegative for some nonnegative operators A and B (cf. [3]).

THEOREM 1. *Let A and B be nonnegative and bounded operators. Then $AB = BA$ if and only if*

$$0 \leq AB^n + B^nA \quad \textit{for } n = 1, 2, \dots.$$

To prove this theorem, we need the following:

LEMMA. *If a projection P satisfies $0 \leq AP + PA$, then $AP = PA$.*

Proof. For arbitrary vectors $x \in P\mathscr{H}$, $y \in (1-P)\mathscr{H}$, and arbitrary complex numbers s and t, we have

$$\begin{aligned} 0 &\leq ((AP + PA)(tx + sy), (tx + sy)) \\ &= 2|t|^2(Ax, x) + 2\operatorname{Re} t\bar{s}(Ax, y), \end{aligned}$$

from which it follows that $0 = (Ax, y)$. Thus we get $AP = PA$.

Proof of Theorem 1. The "only if" part is clear, so we show the "if" part. We may assume that $\|B\| \leq 1$, which means $0 \leq B \leq 1$. Since $0 \leq AB^n + B^nA$, we get

(1) $$0 \leq A\exp(tB) + \exp(tB)A \quad \text{for every } t > 0,$$

from which it follows that

$$0 \leq \exp(-tB)A + A\exp(-tB).$$