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HORIZONTAL PATH SPACES AND CARNOT-CARATHÉODORY METRICS

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In this paper we study a class of sub-spaces of loop spaces which have appeared in the calculus of variations. Generalizing a result of Smale, we show that the space of loops tangent to a distribution satisfying Hörmander's condition is weakly homotopic to the space of all loops. If the distribution is fat, we resolve the end point map from the space of horizontal paths. This resolution has two applications: (1) the proof that the cut-locus on an analytic fat Carnot-Carathéodory manifold is sub-analytic; (2) a study of the singularity of the horizontal loop space. At the end we study the geometry of left-invariant Carnot-Carathéodory metrics on fact nilpotent groups.

0. Introduction. In this paper we will study a class of sub-spaces of loop spaces which have appeared in the calculus of variations, control theorem (cf. [7], [12], [28]).

Let M be a connected manifold, $H \subset TM$ a smooth distribution on M. We say that a H^1-curve $\gamma\colon [0, 1] \to M$ is *horizontal* if it is tangent to H almost everywhere. Let $\langle \cdot, \cdot \rangle$ be a fiberwise metric on H (i.e. a Carnot-Carathéodory metric, or a CC metric for short), then the Carnot-Carathéodory distance (or a CC distance for short) on M is defined to be

$$d(x, y) = \inf E(\gamma)^{1/2}, \qquad \text{where } E(\gamma) = \int_0^1 \left\langle \frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt} \right\rangle dt.$$

Here γ runs over the space of horizontal paths connecting x and y. A classical result of Chow's [9] says that if H satisfies Hörmander's bracket generating condition, then every two points on M can be joined by a horizontal path, so the distance is finite. The interest in horizontal loop spaces in part lies in the fact that they play a similar role in CC metrics as ordinary loop spaces play in Riemannian geometry.

There are many applications of CC metrics to Riemannian geometry, hypoelliptic operators, control theory, physics, etc. see [4], [6], [7], [14], [17], [18], [27], [30], [31], [33], [36], [38], [39], [40].

Let $\Omega_H M(x_0, \cdot)$ (resp. $\Omega_H M(x_0, x_0)$) be the space of H^1-horizontal paths starting from x_0 (resp. H^1-horizontal loops based at x_0).