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BROWNIAN MOTION AND THE HEAT SEMIGROUP ON THE PATH SPACE OF A COMPACT LIE GROUP

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Let G be a compact connected Lie group with identity element e, and let P_eG denote the space of continuous maps $y: [0, 1] \to G$ such that $y(0) = e$. When equipped with the natural group structure and sup metric, P_eG becomes an interesting example of an infinite dimensional nonlinear topological group. The purpose of this paper is to consider certain aspects of analysis on P_eG. Stimulated by a theorem of M. Malliavin and P. Malliavin, we prove the existence of a natural Brownian motion on P_eG which depends only on a choice of bi-invariant metric for G. Our main results, however, concern the heat semigroup associated to the Brownian motion on P_eG. We identify the action of the generator of this semigroup when applied to certain highly regular functions, with a result similar to that obtained earlier by L. Gross in the (linear) abstract Wiener space context.

1. Introduction. Let G be a compact connected Lie group whose identity element we denote e. The purpose of this paper is to construct a natural Brownian motion and associated heat semigroup on the infinite dimensional nonlinear space of continuous maps $y: [0, 1] \to G$ such that $y(0) = e$. We refer to this space as P_eG. The Brownian motion on P_eG depends only on a choice of bi-invariant metric for G.

Note that P_eG inherits a group structure from G: for $y_1, y_2 \in P_eG$ define y_1y_2 by $(y_1y_2)(t) = y_1(t)y_2(t)$. The constant path at the identity is the identity element in P_eG. Given a Riemannian metric g on G, let $d_g(\cdot, \cdot)$ denote the associated distance function on $G \times G$, which induces a metric on P_eG given by $\sup_{t\in[0,1]} d_g(y_1(t), y_2(t))$. In any such metric P_eG becomes a Polish topological group. This structure leads to a convolution law for probability measures on the Borel field of P_eG. In the special case where the metric g is bi-invariant on G, it happens that the associated bi-invariant Wiener measures on P_eG form a convolution semigroup. This fact, discovered by M. Malliavin and P. Malliavin [**16**], is the origin of Brownian motion on P_eG. Lemma 2.2 of this paper supplies the additional required estimate for continuity of sample paths. We also provide an elementary analytic proof of the Malliavins' theorem in Lemma 2.1.